University of Maryland

ENES 106:
Single Variable Calculus

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Contents

0 Introduction

There are essentially two components to doing calculus. The first component is computational - learning all of the tricks and formulas that make up the bulk of the "work" for any given problem. The second component is intuition and visualization - a feeling for the infinitesimal, of what it means to be "approaching" a limit or adding up infinitely many pieces of infinitely small area to obtain something meaningful. Both aspects of the subject are crucial. Intuition without the computational firepower to back it up won't get very far. For example the "intuitive" statement:

Theorem 0.1. *A continuously differentiable function's minimum occurs either at its endpoints or where its derivative is zero.*

is very nice in that it doesn't appear to involve any formulas, but it doesn't help too much if one can't then compute the function's derivative and solve the resulting algebraic equation to obtain its roots.

On the flip side, it is entirely possible to get lost in computation when an intuitive approach would save you a lot of trouble. For example consider the following improper integral, that asks you to find the (signed) area underneath the curve $f(x) = e^{-x^2}x^3$ between $-\infty$ and ∞:

$$\int_{-\infty}^{\infty} e^{-x^2}x^3 dx \tag{0.1}$$

This is a Hard Integral™if one immediately attacks it computationally, but if one thinks about the geometry of the curve and what doing an integral actually means the answer just falls out. The function $f(x) = e^{-x^2}x^3$ is what's called an "odd function," meaning simply that its values to the left of zero are the negative versions of its values to the right of zero:

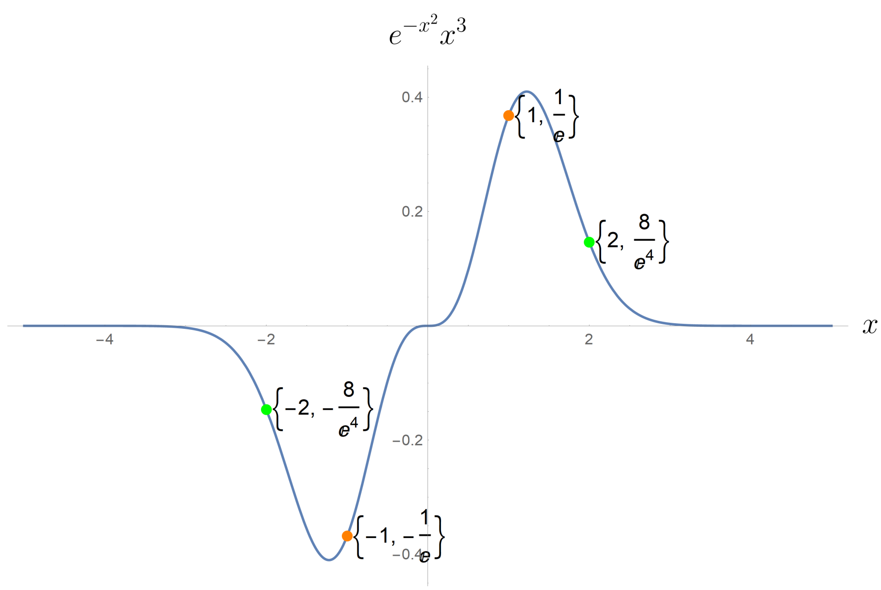


Figure 1: Another way to the think about the fact that $f(x) = e^{-x^2}x^3$ is an odd function is that it stays the same if you rotate it by 180 degrees.

Doing an integral in this case means computing the signed area between the curve and the x axis (positive if the area is above the x axis, negative if the area is below the x axis). But since we are integrating from $-\infty$ to ∞ and the function is odd, the area to the right of the origin (positive) and the area to the left of the origin (negative) are equal in magnitude but opposite in sign, and hence cancel out. Thus intuition gets us to

$$\int_{-\infty}^{\infty} e^{-x^2} x^3 dx = 0 \tag{0.2}$$

This is an important truth about doing calculus - doing calculus is not the same as doing calculations, and while sometimes calculation is unavoidable, other times it is best avoided!

0.1 Course Schedule

This is a five week whirlwind tour of calculus. The rough outline for the course will be as follows:

- June 27 - July 1

June 27	Tuesday	Review: Domain and Range, Linear Functions, Exponential Functions
June 28	Wednesday	Review: Transformations of Functions, Inverse Functions, Logarithmic Functions
June 30	Thursday	Review: Rational Functions, Trig Functions
July 1	Friday	Limits, Continuity, $\epsilon - \delta$ Calculus

- July 4 - July 8

July 4	Monday	Holiday
July 5	Tuesday	Tangent Lines, Velocities, and the Derivative
July 6	Wednesday	Derivatives of polynomials, exponentials, products, and quotients
July 7	Thursday	Derivatives of composite functions, inverse functions, and Implicit Differentiation
July 8	Friday	Applications of Differentiation: Approximating Functions, L'Hopital's Rule

- July 11 - July 15

July 11	Monday	Applications of Differentiation: Optimization
July 12	Tuesday	Applications of Differentiation: Exponential growth, Related Rates
July 13	Wednesday	Integration: Definite Integrals, The Riemann Integral
July 14	Thursday	Integration: Indefinite Integrals, The FTCs, Some Anti-Derivatives
July 15	Friday	Integration Techniques: Integration by substitution

- July 18 - July 22

July 18	Monday	Integration Techniques: Integration by parts
July 19	Tuesday	Trigonometric Integrals
July 20	Wednesday	Partial Fractions Integration
July 21	Thursday	Applications of Integration: Volumes, Center of Mass, Differential Equations
July 22	Friday	Sequences, Convergence/Divergence, Geometric Series, Absolute Convergence

- July 25 - July 29

July 25	Monday	Series Convergence Tests: The Comparison Test, The Limit Comparison Test
July 26	Tuesday	Series Convergence Tests: The Ratio Test, The Root Test, The Integral Test
July 27	Wednesday	Power Series and the Interval of Convergence
July 28	Thursday	Taylor Series, Applications and Examples.
July 29	Friday	A beautiful result: DeMoivre's Formula and Euler's Identity

0.2 Course Format and Materials

This course relies heavily on two textbooks:

- Calculus With Concepts by Denny Gulick and Robert Ellis.
- Calculus Single and Multi Variable by Deborah Hughes-Hallet et al.

You don't need to buy either of these books. The first book, Calculus With Concepts, is available via the course Web Assign page (add link once webassign is setup). I will always post course notes to this document in advance of class and I will provide problems separately (some from the books, some not). However, if you want even more practice material and background these are both terrific books.
The breakdown of in class time will be as follows:

- 25 minutes: Lecture / exposition by me.
- 65 minutes Problem solving in groups.
- 30 minutes Problem solving presentations / exposition by you.

There is a course discord https://discord.gg/k4s3FD6b8j where you can and should ask each other and me questions. I am also always reachable at cbartondock@gmail.com or cdock@umd.edu

The undergraduate mentor for the course is Kenan Asmerom Atlaw, who can be reached at katlaw@umd.edu.

1 Functions and a review of some precalculus topics

1.1 Domain and range

Resources

- Khan Academy on Functions
- Calculus with Concepts 1.3
- Single and Multi-Variable Calculus 1.1

A function f is a rule that assigns each element of of one set (collection of objects) X to a single element of a second set Y. We write $f : X \to Y$.

Example 1.1. The following is a function from the set $X = \{\text{Red}, \text{Green}, \text{Blue}\}$ to the set $Y = \{\text{Purple}, \text{Yellow}, \text{Orange}\}$:

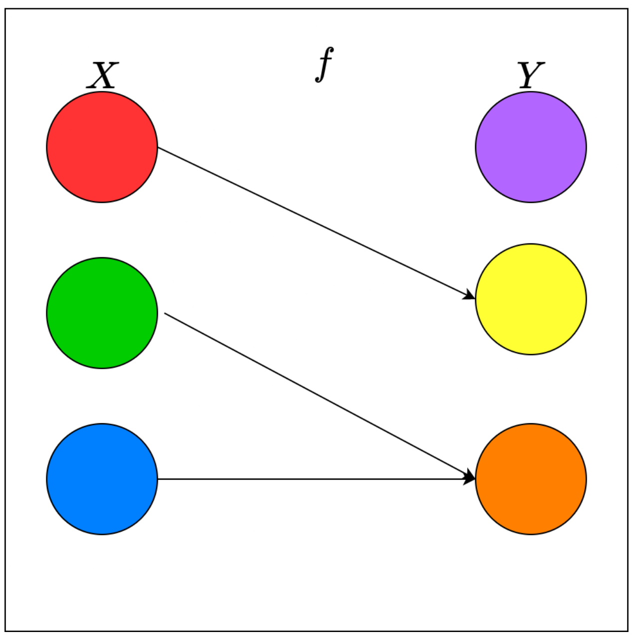


Figure 2: A function doesn't have to act on numbers. It is just a machine that takes inputs and spits out outputs. It can act on integers, real numbers, colors, whatever!

The set X is called the domain of the function f (where its inputs live) and the set Y is usually called its co-domain (where its outputs live). Meanwhile, the actual values that the function outputs are called *the range* of the function (usually written $f(X)$). In this case the range is $f(X) = \{\text{Yellow}, \text{Orange}\}$ (purple isn't "hit" by the function f).

Note that a function is not guaranteed to hit every element of the co-domain, and must assign each element of the domain *to only one element* of the co-domain.

Non-Example 1.1. The following association *is not* a function:

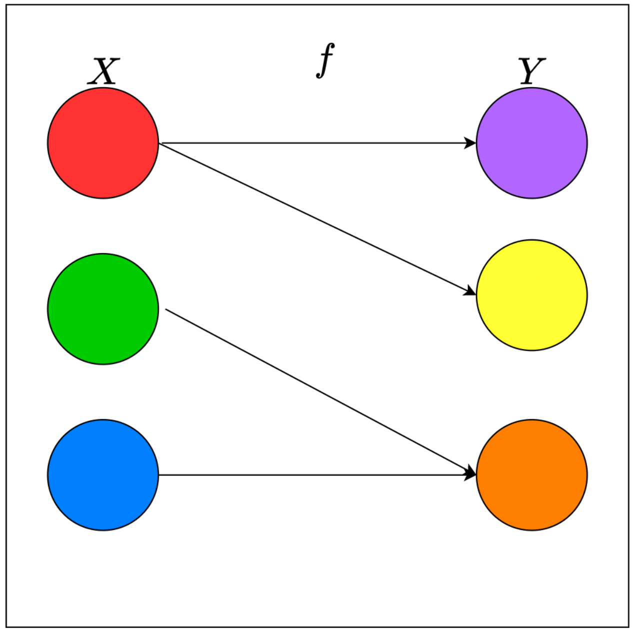


Figure 3: A function must assign each input only to a single output, this is not a function!

Usually (and from now on in this course) the inputs and outputs of functions will be real numbers, elements of the set $\mathbb{R}$ (the real number line):

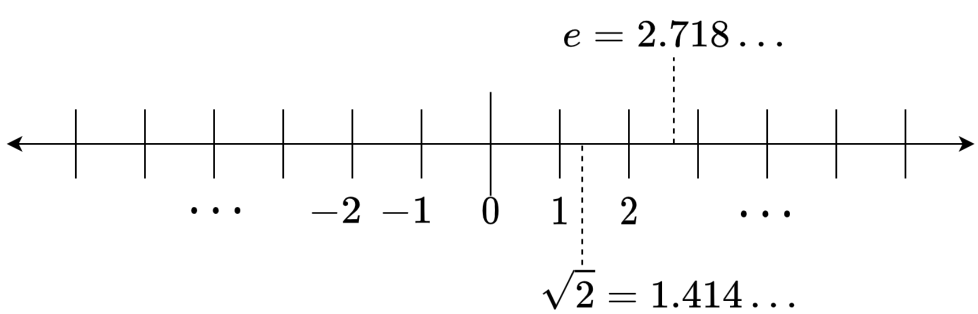


Figure 4: The real number line $\mathbb{R}$

Example 1.2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$, the humble parabola. Unlike the color function above (which had three inputs), this function has too many inputs to list. We can list a few:

x	-2	-1	0	0.5	$\pi = 3.14\ldots$	$\cdots$
f(x)	4	1	0	0.25	$\pi^2 = 9.86\ldots$	$\cdots$

Instead of listing all the inputs and outputs of a function of a real variable, what we usually do is graph the collection of points $(x, f(x))$ such that x is a real number (the input of the function is the x coordinate and the output of the function is the y coordinate). In this case we get the graph:

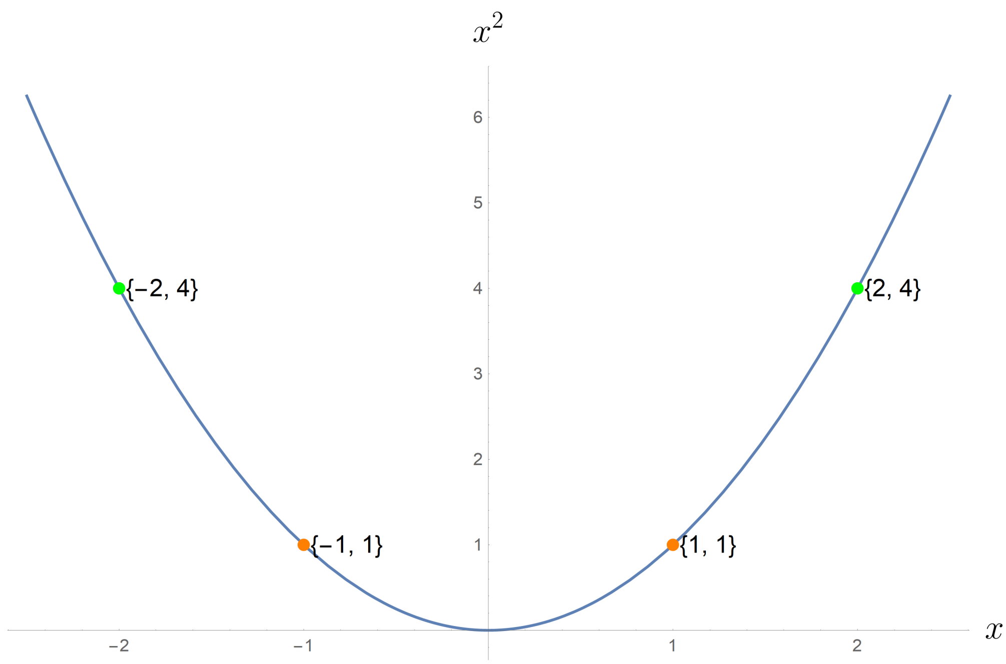


Figure 5: The graph of $f(x) = x^2$, a parabola.

The domain of this function $x \mapsto x^2$ is all real numbers, whereas its range is only the non-negative real numbers $[0, \infty)$.

Given two functions $f, g : \mathbb{R} \to \mathbb{R}$ we can form new functions in several ways:

- Functions can be added and subtracted: $(f + g)(x) = f(x) + g(x)$ and $(f - g)(x) = f(x) - g(x)$.

- Functions can be multiplied: $(f \cdot g)(x) = f(x)g(x)$.

- Functions can be divided: $(f/g)(x) = f(x)/g(x)$. The new functions f/g must exclude from its domain all points where $g(x) = 0$.

- Functions can be *composed*, $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$. Note that composition *is not* commutative, for example if $f(x) = x^2$ and $g(x) = x + 1$ then $g(f(x)) = x^2 + 1$ and $f(g(x)) = (x + 1)^2 = x^2 + 2x + 1$.

What about functions that don't make sense for arbitrary real numbers, like $f(x) = \frac{1}{x}$ or $f(x) = \sqrt{x}$?

Example 1.3. Consider $f(x) = \frac{1}{x}$. Clearly this function is fine and has meaningful outputs for all inputs except $x = 0$, but it has a serious problem at $x = 0$: division by zero is Not Allowed™. The way we deal with this is to restrict the domain of the function to be $(-\infty, 0) \cup (0, \infty)$. The symbol $\cup$ here means the union of the two intervals, that is to say the domain of the function is all numbers that are either less than zero or bigger than zero). Incidentally, the range of this function is also $(-\infty, 0) \cup (0, \infty)$. Its graph is:

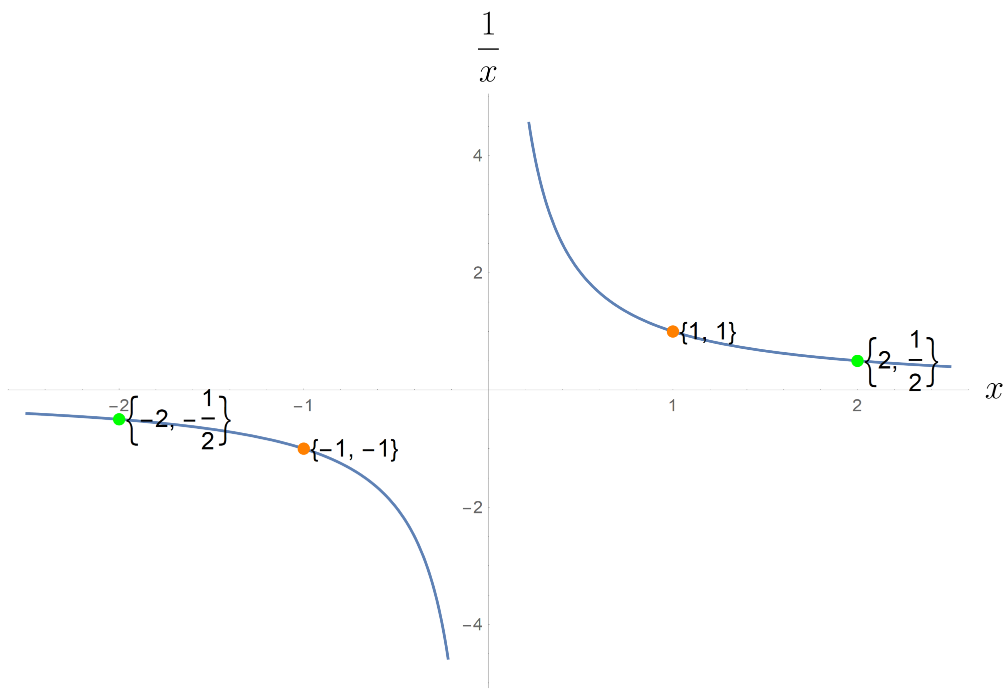


Figure 6: The graph of $f(x) = 1/x$, a hyperbola.

Example 1.4. The function $f(x) = \sqrt{x}$ (when considered to have real inputs and outputs) only makes sense for non-negative inputs, thus it has domain $[0, \infty)$.

It is of course possible to find the domain of more complicated functions by systematically removing all of the points and intervals for which they are not defined, for example:

Example 1.5. Find the domain of the function

$$g(x) = \frac{\sqrt{100 - x}}{(x-2)(\sqrt{x} - 5)} \tag{1.1}$$

The numerator is well defined so long as $x \leq 100$ (because of the square root), and the denominator is well defined so long as $x \geq 0$ (because of the square root), $x \neq 2$ (because of the factor $(x-2)$), and $x \neq 25$ (because of the factor $(\sqrt{x} - 5)$). Thus the domain of g is $[0, 100] \setminus \{2, 25\} = [0, 2) \cup (2, 25) \cup (25, 100]$. The symbol $\setminus$ here is the "set minus", thus in English the domain of g is the closed interval from 0 to 100 excluding the numbers 2 and 25.

In general if f and g are functions then:

- The domain of the sum $f + g$ and of the product $f \cdot g$ is:

$$\text{Domain}[f + g] = \text{Domain}[f \cdot g] = \text{Domain}[f] \cap \text{Domain}[g] \tag{1.2}$$

- The domain of the quotient f/g is:

$$\text{Domain}[f/g] = \text{Domain}[f \cdot g] = \text{Domain}[f] \cap \text{Domain}[g] \setminus \{x | g(x) = 0\} \tag{1.3}$$

- The domain of the composition $f \circ g$ is:

$$\text{Domain}[f \circ g] = \text{Domain}[g] \cap g^{-1}(\text{Domain}[f]) \tag{1.4}$$

Where if A is a set then $g^{-1}(A) = \{x | g(x) \text{ is in } A\}$. Intuitively, we exclude all of the "problem points" for g plus all of the points that get sent to problem points for f by g.

It is possible to test whether a given graph comes from a function using the "vertical line test," which asks whether it is possible to pass a vertical line through the curve at more than one point. If so, then the curve is not a function:

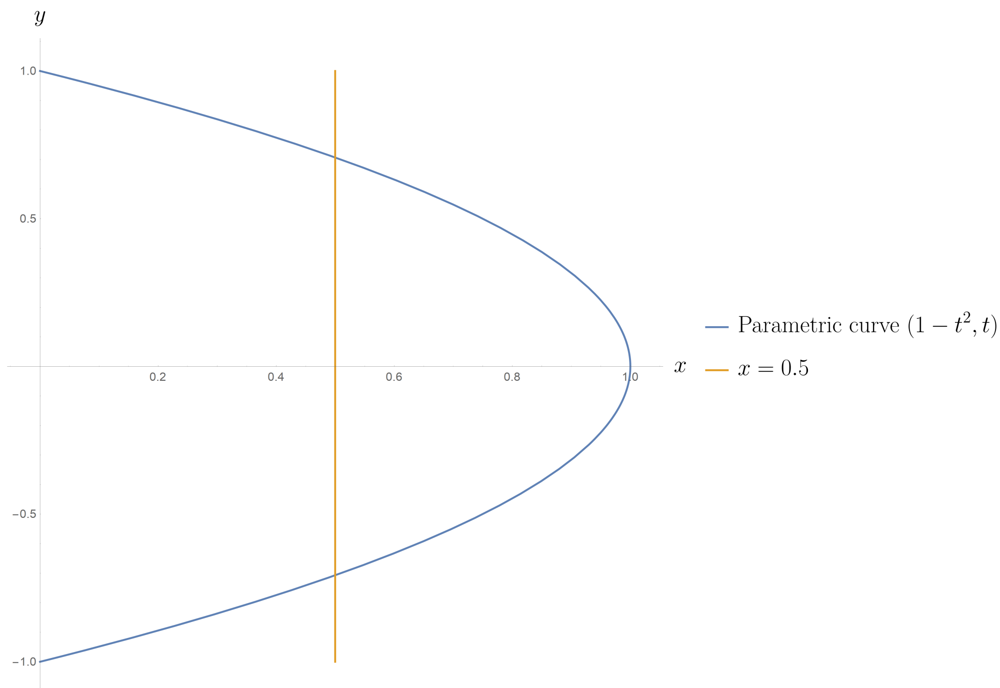


Figure 7: The curve shown here is not a function of x, because it would purport to assign (for example) $x = 0.5$ to two values.

1.2 Linear functions

Resources

- Khan Academy on Linear Equations
- Calculus with Concepts 1.2
- Single and Multi-Variable Calculus 1.1

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is called linear if there exists a constant m so that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = m \tag{1.5}$$

for all inputs x_1 and x_2. This constant m is known as *the slope* of the linear function, because it encodes how much the function changes in proportion to its inputs.

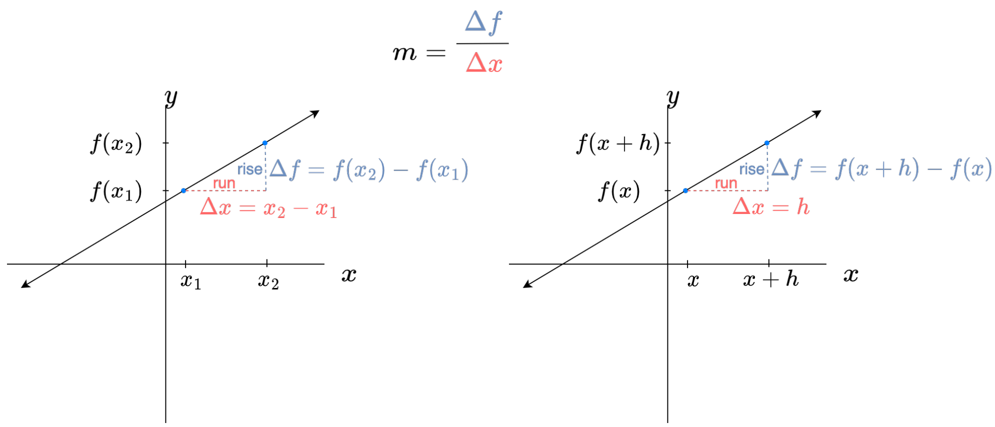


Figure 8: The geometric meaning of (1.5) and (1.6)

The equation (1.5) is also often written as

$$\frac{\Delta f}{\Delta x} = \frac{f(x+h) - f(x)}{h} = m \tag{1.6}$$

The "rise" $\Delta f = f(x+h) - f(x)$ is the amount of change in f when the input changes by the "run" $\Delta x = (x+h) - h = h$. Note that if $f(0) = b$, that is to say the y intercept of f is b, then (setting $x_2 = x$ and $x_1 = 0$ and solving (1.5) for $f(x)$) we find

$$f(x) = mx + b \tag{1.7}$$

The standard form for linear equations with finite slope.

Linear trends are ubiquitous in data, when a given quantity increases or decreases by (roughly) the same amount in every fixed period of time. In [HH+02] they give the example of the pole vaulting world record height between 1900 and 1912:

Year	1900	1904	1908	1912
Height (in)	130	138	146	154

The trend is clear – the winning height increases by about 8 inches every 4 years (obviously it was not possible for this trend to continue indefinitely!). This means that the approximating linear function should have a slope of 2 (rise is 8 and run is 4) and be equal to 130 when $t = 1900$, thus if we plot the data on top of $h(t) = 2(t - 1900) + 130$ we get:

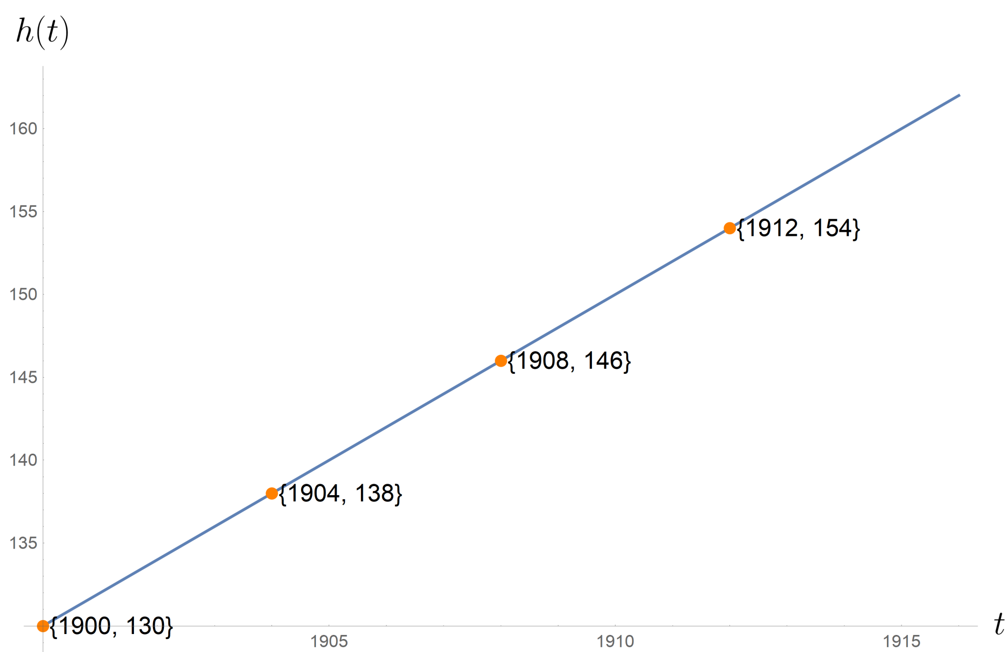


Figure 9: The graph of the linear function $h(t) = 2(t - 1900) + 130$, overlaid with the data from the pole vaulters.

Intuitively, a function is non-linear if its graph is not a straight-line. None of the previously mentioned functions $f(x) = x^2$, $f(x) = \sqrt{x}$, or $f(x) = 1/x$ are linear. This can be confirmed by computing a few values of the rise over run ratio (1.5) at different points and observing that the rise over run ratio changes as x_1 and x_2 vary.

Linear functions may be simple, but they form the bedrock of differential calculus. In particular, we will be quite interested in *secant lines*:

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$ and two numbers a and b, the secant line of f from a to b is the line through the point $A = (a, f(a))$ and the point $B = (b, f(b))$. That is, the line

$$\frac{f(x) - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \iff f(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \tag{1.8}$$

Example 1.6. We obtain the secant line of $\sin(x)$ between 0 and $\frac{\pi}{4}$. Firstly the slope of the secant line is

$$\frac{\sin(\pi/4) - \sin(0)}{\pi/4 - 0} = \frac{1/\sqrt{2}}{\pi/4} = \frac{2\sqrt{2}}{\pi} \approx .9 \tag{1.9}$$

Secondly we know that $\sin(0) = 0$, thus from (1.8) we get that the secant line is $f(x) = (2\sqrt{2}/\pi)x$. Plotting both $\sin(x)$ and $f(x) = (2\sqrt{2}/\pi)x$ yields:

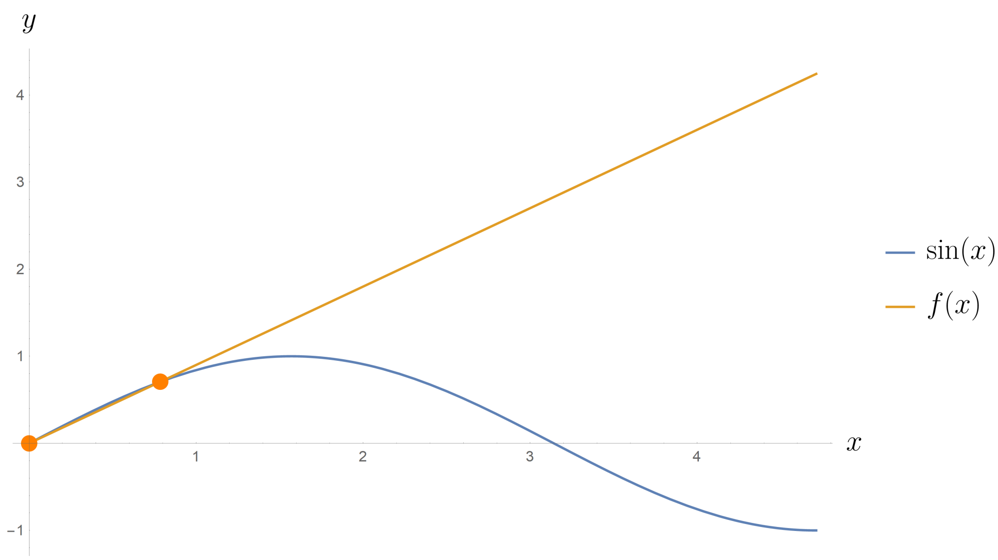


Figure 10: The secant line of $\sin(x)$ from $x = 0$ to $x = \frac{\pi}{4}$. Notice that the secant line appears to approximate the tangent line of $\sin(x)$ at zero, and that this approximation should get better as the second secant point gets closer to zero.

1.3 Exponential functions

Resources

- Khan Academy on Exponential Growth and Decay
- 3Blue1Brown Exponential Growth and Pandemics
- Calculus with Concepts 1.2
- Single and Multi-Variable Calculus 1.8

Exponential functions are somewhat similar in character to linear functions, except that instead of having a constant *difference* for a given difference in inputs, they have a constant *ratio* for a given difference in inputs:

Definition. A function f is exponential if there exists a constant r such that

$$(f(x+h)/f(x))^{1/h} = r \tag{1.10}$$

for all x and h.

Note the similarity of (1.10) to (1.6) – replace subtraction by division and division by root taking. The constant r is known as the exponential growth rate of the function if $r > 1$ and the exponential decay rate if $r < 1$ (if $r = 1$ the function f is constant).

Consider the following data on bacteria populations in a petri-dish (populations without natural predators exhibit exponential growth until they run into resource constraints):

Time (in days)	0	1	2	3	4
Population (in thousands)	5	10	20	40	80

The population is doubling every day, with an initial population of 5000 individuals. Thus the exponential curve that models the population growth is

$$p(t) = 5000 \cdot 2^t \tag{1.11}$$

Note that $p(t+h)/p(t) = 2^h$, indicating an exponential growth rate of 2. We can plot both $p(t)$ and the data to obtain

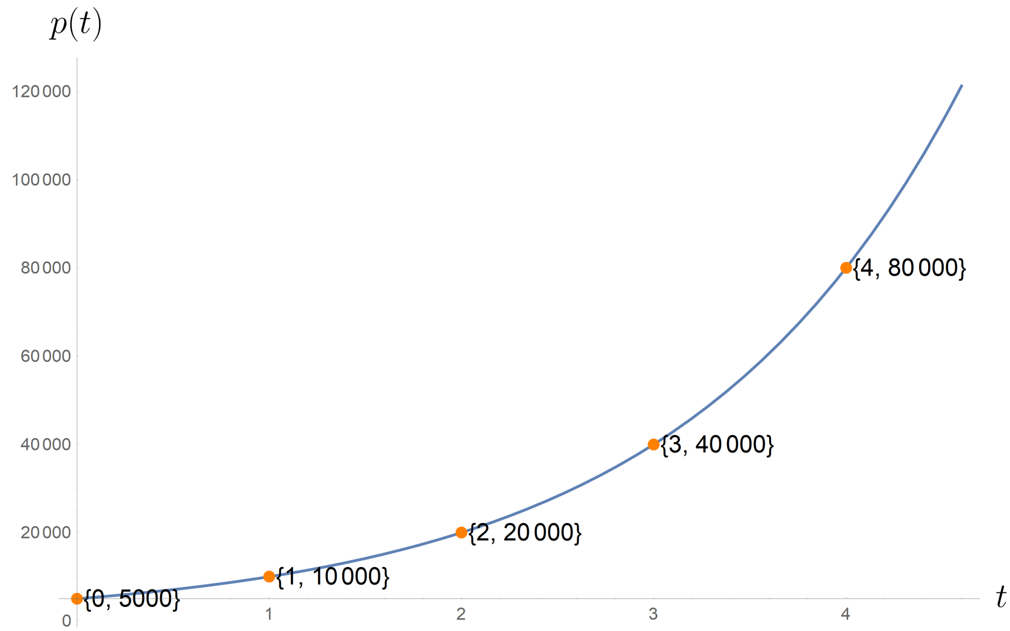


Figure 11: A plot of the curve $p(t)$ alongside the data above.

Similarly to linear functions, exponential functions are determined once the exponential rate parameter r is known and the value of the function is known at any particular point. Indeed, suppose $f(0) = A$. Then (1.10) gives

$$f(0+h)/f(0) = r^h \iff f(h) = A \cdot r^h \tag{1.12}$$

Thus the general form for an exponential function is $f(x) = A \cdot r^x$. The number r is also known as the *base* of the exponential. The most common and useful base for the exponential is actually the famous irrational number $e = 2.718281828459\ldots$, but unfortunately the reason this base is so useful will have to wait until we have a little bit more of the machinery of calculus available to us.

Note that we can change base using logarithms. Indeed if $f(x) = Ar^x$ and we want to know the form for f with exponential base 2 then we note that $r = 2^{\log_2(r)} = 2^{\ln(r)/\ln(2)}$. Thus $f(x) = A \cdot 2^{(\ln(r)/\ln(2))x}$. This is useful for obtaining *doubling times* or *half-lives*. If an exponential function is in the form

$$f(t) = A \cdot 2^{t/T} \tag{1.13}$$

then it doubles every T units of time (every time t increases by T it picks up an extra factor of 2). Thus T is called the doubling time of the exponential function. If the exponential function is given instead via $f(t) = Ar^t$ with some rate r then we can still obtain the doubling time since

$$Ar^t = A \cdot 2^{(\ln(r)/\ln(2))t} = A \cdot 2^{t/(\ln(2)/\ln(r))} \tag{1.14}$$

Thus the doubling time is $T = \ln(2)/\ln(r)$.

Similarly, if an exponential function is in the form

$$f(t) = A \cdot 2^{-t/\tau} = A(1/2)^{t/\tau} \tag{1.15}$$

then it decays by a factor of $1/2$ every τ units of time (every time t increases by τ it picks up an extra factor of $\frac{1}{2}$), thus τ is referred to as the half-life. This type of equation is often used to model the decay of radio-active isotopes. For example, one of the more dangerous isotopes released into the environment by the Chernobyl nuclear disaster in 1986 was Cs-137 or Caesium 137, which has a half life of $\tau \approx 30$ years. About 25 kilograms was released into the environment, so if we want to figure out how much is still there we can compute it via $f(2022)$ where $f(t)$ is

$$f(t) = 25 \cdot 2^{-(t-1986)/30} \tag{1.16}$$

$$f(2022) \approx 10.9 \tag{1.17}$$

Thus there are still approximately 11 kilograms of Cs-137 remaining from the Chernobyl disaster (this might not seem like much, but it and Strontium 90 are the main reasons that the area will not be safe for human habitation for at least another 300 years or so).

One particularly useful financial application of exponential functions is interest, and specifically compounding interest. If an investment initial investment I_0 grows with a rate r per year (a typical r might be .02 or 2%) then after the first year one has

$$I_1 = I_0 \cdot 1.02 = I_0(1+r) \tag{1.18}$$

If one then re-invests, the interest rate applies to all of I_1 not just I_0, so that after the second year one has

$$I_2 = I_1 \cdot 1.02 = I_0(1.02)^2 = I_0(1+r)^2 \tag{1.19}$$

This process continues, and so after t years one has

$$I(t) = I_0(1+r)^t \tag{1.20}$$

So far so good, but then you realize you could actually make more on your investment by re-investing earlier. Why wait until the end of the year to put the profit back into the investment and start receiving interest on it? So you decide to re-invest k times per year (for example $k = 12$ would be monthly compounding). In this case the return for each "compounding period" is r/k, but there are k such periods, so after one year you would have

$$I_1 = I_0 \underbrace{(1+\frac{r}{k})\cdots(1+\frac{r}{k})}_{k \text{ factors}} = (1+\frac{r}{k})^k \tag{1.21}$$

And as before after two years

$$I_2 = I_1(1+\frac{r}{k})^k = I_0(1+\frac{r}{k})^{2k} \tag{1.22}$$

So that finally after t years one has

$$I(t) = I_0(1+\frac{r}{k})^{kt} \tag{1.23}$$

This is the formula for the return on an investment I_0 with rate r and compounding k times per unit of time (typically years). One might well ask what happens as we compound more and more frequently. Obviously profit increases, but does it do so indefinitely? Unfortunately not, but we will have to wait until we have learned about limits to answer this question in full.

Working with exponential functions requires one to have a full mastery of the properties of exponentiation. For reference, recall that for any $a > 0$ and x, y real numbers:

- Exponential of a sum is *the product* of exponentials:

$$a^{x+y} = a^x a^y = a^y a^x \tag{1.24}$$

- Exponential of a product is *the composition* of exponentials (in any order):

$$a^{xy} = (a^x)^y = (a^y)^x \tag{1.25}$$

  Note that this is not the same thing as $a^{(x^y)}$ or $a^{(y^x)}$. Exponentiation is *not associative*!

- The exponential of a negation is the reciprocal of the exponential:

$$a^{-x} = \frac{1}{a^x} \tag{1.26}$$

1.4 Transformations and Symmetries of Functions

Resources

- Khan Academy on Transformations and Symmetries of Functions
- Calculus with Concepts 1.5
- Single and Multi-Variable Calculus 1.3

Given a function $f : \mathbb{R} \to \mathbb{R}$ we can obtain several families of functions by shifting and scaling f. For the following examples let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1 - x^2$. The graph of f is an upside down parabola with its highest point at $(0, 1)$ and with x-intercepts ± 1. Let $a > 0$

Example 1.7 (Vertical Shift)**.** We can shift a function f up by a units by letting $f(x) \to f(x) + a$, or down by a units by letting $f(x) \to f(x) - a$:

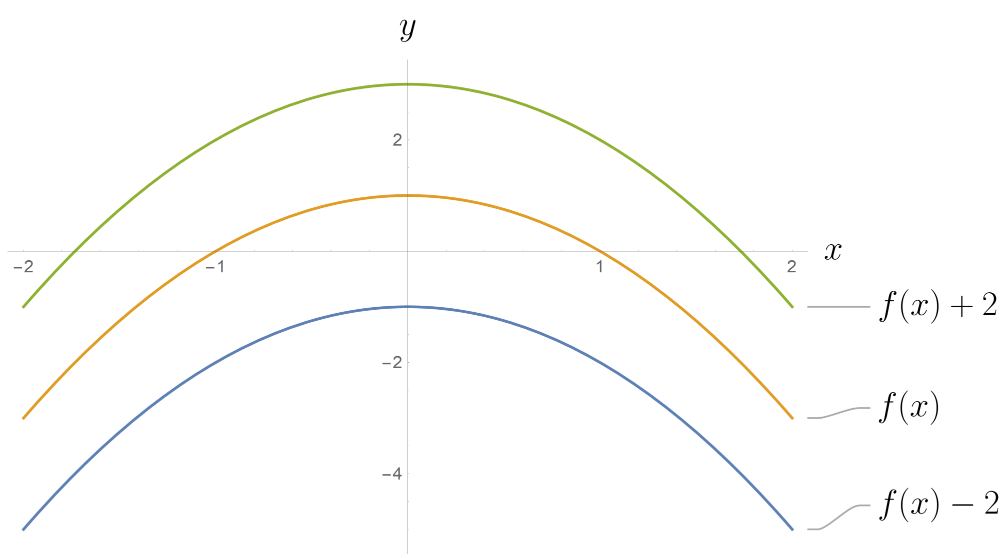


Figure 12: Shifting f up and down by 2 units.

Example 1.8 (Horizontal Shift)**.** We can shift a function f to the right by a units by letting $f(x) \to f(x-a)$, or to the left by a units by letting $f(x) \to f(x+a)$:

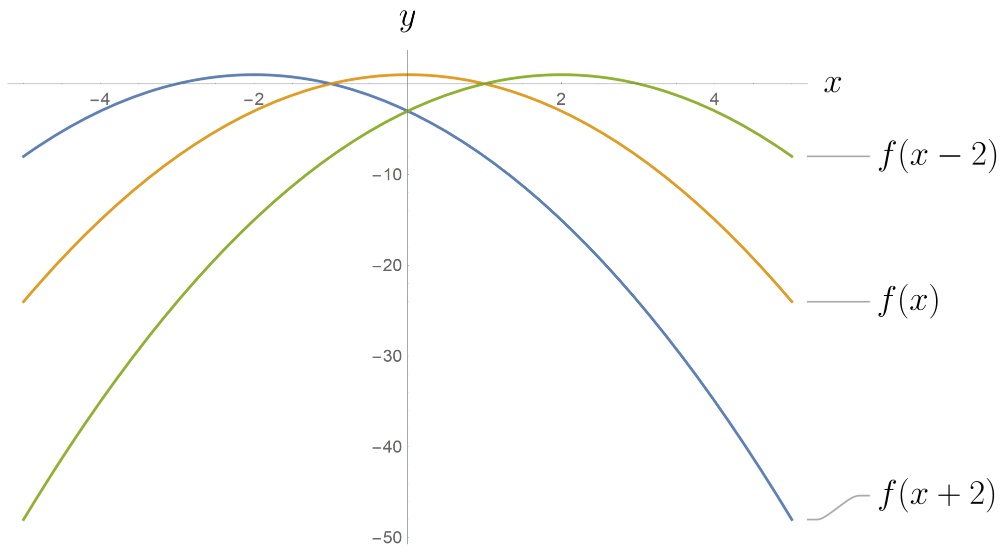


Figure 13: Shifting f to the left and right by 2 units.

The fact that $f(x) \to f(x-2)$ shifts the function f to the right rather than the left is sometimes counter-intuitive to students. It can help to think of $f(x) \to f(x-2)$ pulling the function values from the left to the right, e.g. the value at 3 gets replaced by the value at 1 and the value at 1 gets replaced by the value at -1, etc.

Example 1.9 (Vertical Scale)**.** We can stretch a function f in the vertical direction by letting $f(x) \to af(x)$:

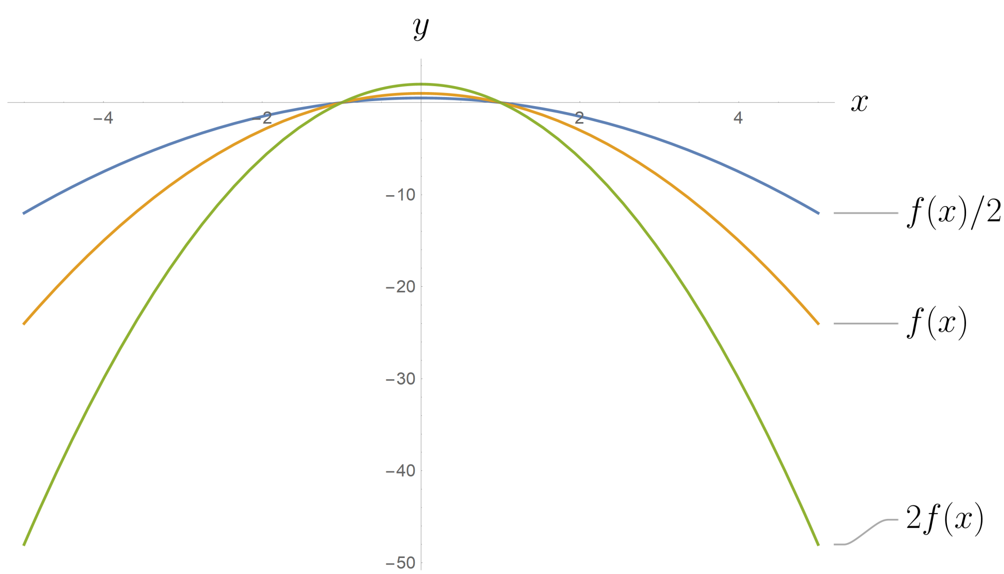


Figure 14: Stretching and squishing f vertically by a factor of 2. If $a > 1$ the function is stretched, if $a < 1$ the function is squished.

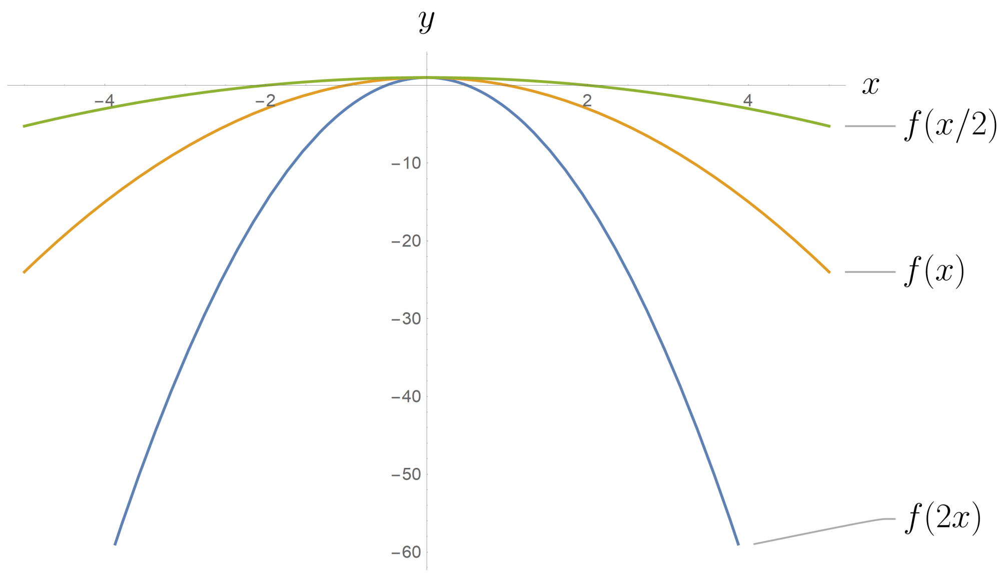


Figure 15: Stretching and squishing f horizontally by a factor of 2. If $a > 1$ the function is stretched, if $a < 1$ the function is squished.

Example 1.10 (Horizontal Scale)**.** We can stretch a function in the horizontal direction by letting $f(x) \to f(x/a)$. As in the case of horizontal translation, it can help to think of the horizontal stretch $f(x) \to f(x/a)$ as pulling values from close to the origin to replace values further from the origin. For example if $f(x) \to f(x/2)$ then $f(2)$ is replaced by $f(1)$, $f(-2)$ is replaced by $f(-1)$, etc.

Example 1.11 (Reflection over the y axis)**.** The operation $f(x) \to f(-x)$ reflects the function f over the y-axis:

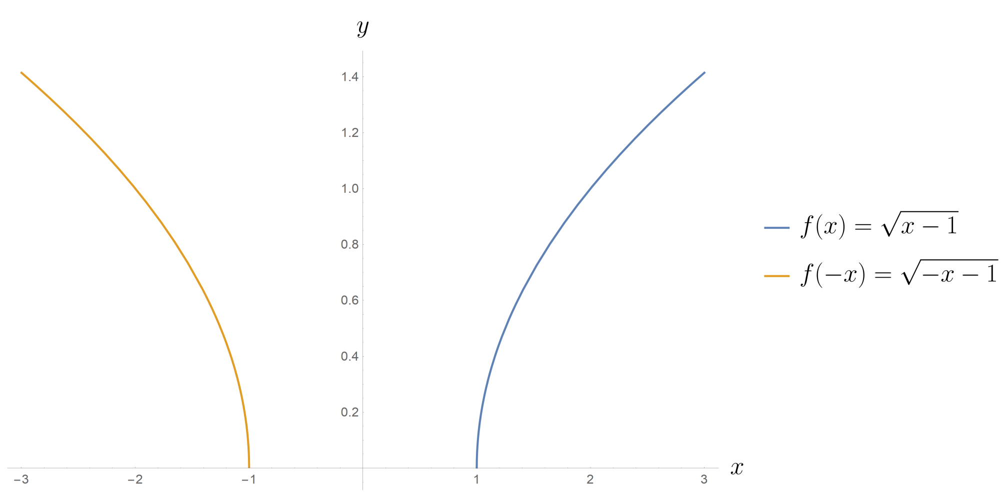


Figure 16: The function $f(x) = \sqrt{x-1}$ reflected over the y axis

We call a function *even* if it is unchanged by reflecting it over the y-axis, that is if $f(x) = f(-x)$. The reason for this terminology is that a polynomial function is even if and only if it contains only even powers, e.g. x^2, $1 - x^2$, $x^4 - x^2$, etc.

Example 1.12 (Reflection over the origin / Rotation by π)**.** The operation $f(x) \to -f(-x)$ reflects the function f over the origin, or equivalently it rotates the function by π radians:

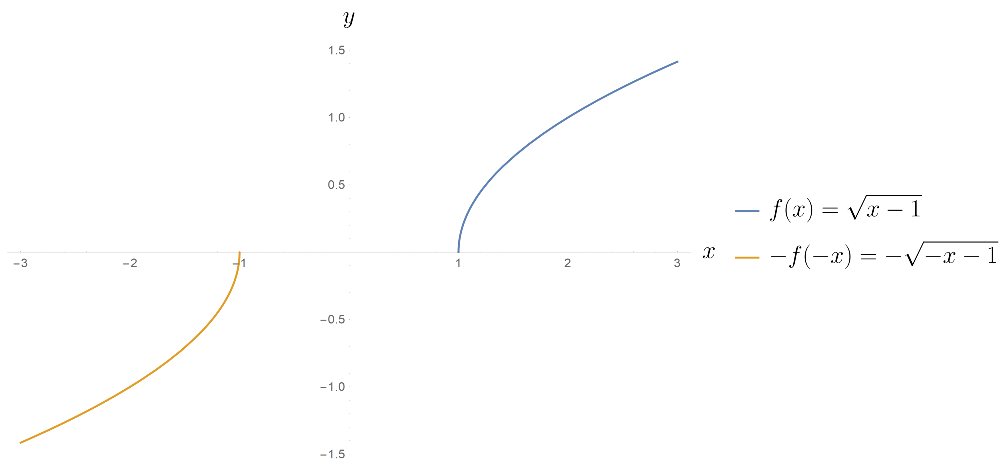


Figure 17: The function $f(x) = \sqrt{x-1}$ reflected over the origin

We call a function *odd* if it is unchanged by reflecting it over the origin, that is if $f(x) = -f(-x)$ or $f(-x) = -f(x)$. The reason for this terminology is that a polynomial function is odd if and only if it contains only odd powers, e.g. x, x^3, $x^5 - x$, etc.

Example 1.13 (Reflection over the x axis)**.** The operation $f(x) \to -f(x)$ reflects f over the x axis:

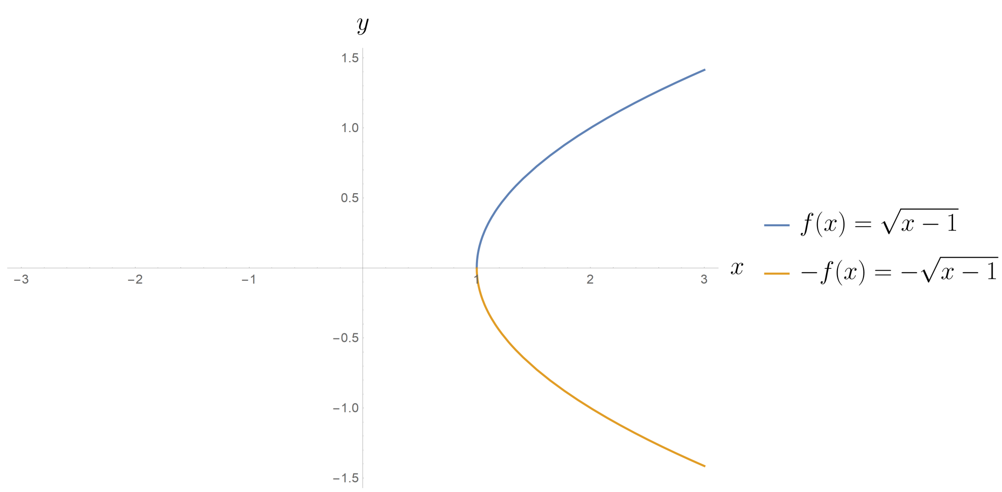


Figure 18: The function $f(x) = \sqrt{x-1}$ reflected over the x axis

Note that the only *function* that is symmetric over the x axis, i.e. that satisfies $f(x) = -f(x)$, is $f(x) = 0$ (other curves may satisfy this symmetry, like the circle, but they are not functions).

1.5 Inverse functions

Resources

- Khan Academy on Inverse Functions and Compositions
- Calculus with Concepts 7.1
- Single and Mult-Variable Calculus 1.3

The inverse (sometimes called the left inverse) of a function $f : X \to Y$ is a second function $f^{-1} : f(X) \to X$ such that $f^{-1}(f(x)) = x$. For example if the function $g : \{\text{Red}, \text{Green}, \text{Blue}\} \to \{\text{Purple}, \text{Yellow}, \text{Orange}\}$ is given as on the left then its inverse is given as on the right:

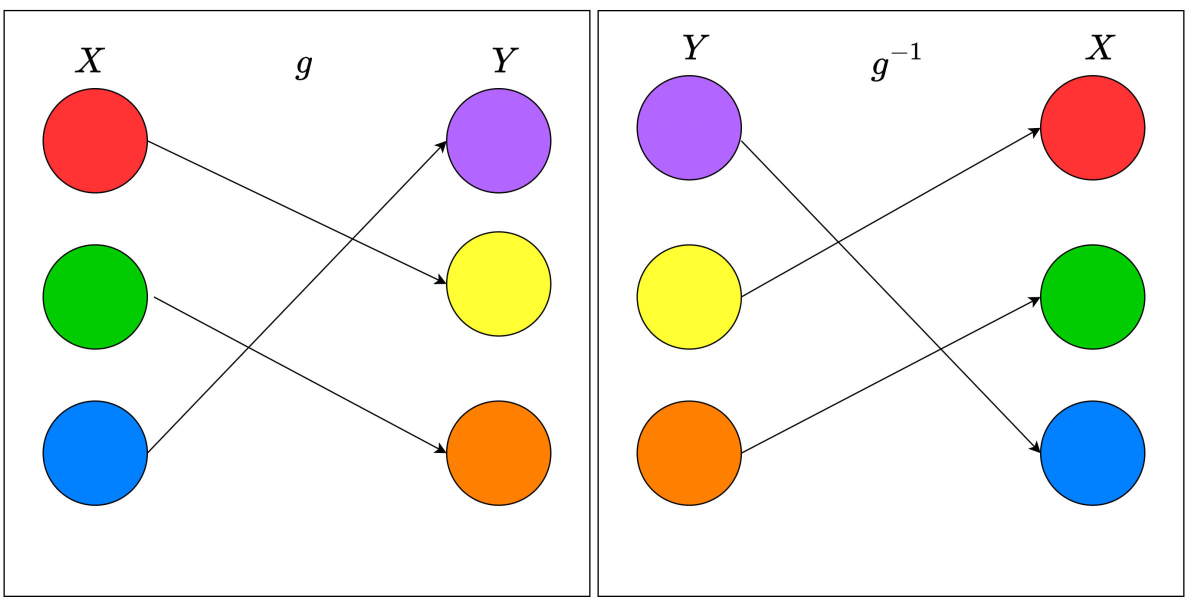


Figure 19: An invertible function $g : X \to Y$ and its inverse $g^{-1} : Y \to X$. Notice that if you follow any one of the arrows you end up back where you start, e.g. Red goes to Yellow via g and then Yellow goes to Red via g^{-1}.

Not all functions have inverses, for example, suppose g were instead given as:

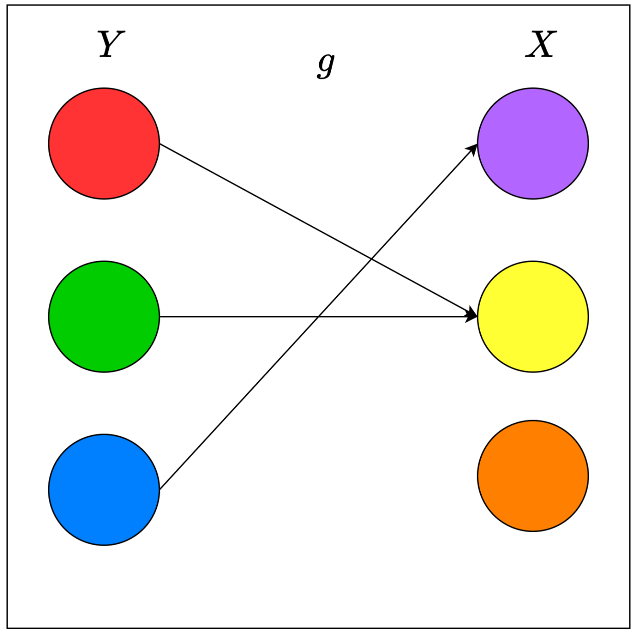


Figure 20: A non invertible function.

In this case there is no inverse function g^{-1} because "information is lost" when both Red and Green are sent to Yellow. Any inverse function would have to assign Yellow to both Red and Green, but functions can only take a single value for a given input. We call such a function non-invertible. In general a function $g : X \to Y$ is invertible (also known as injective) if no two elements of X get sent to the same element of Y. Rephrasing that slightly, a function $g : X \to Y$ is non-invertible if we can find x_1 and x_2 that are different from each-other and such that $g(x_1) = g(x_2)$.

For a function $f : \mathbb{R} \to \mathbb{R}$, we can check invertibility by seeing if the function passes the "horizontal line test":

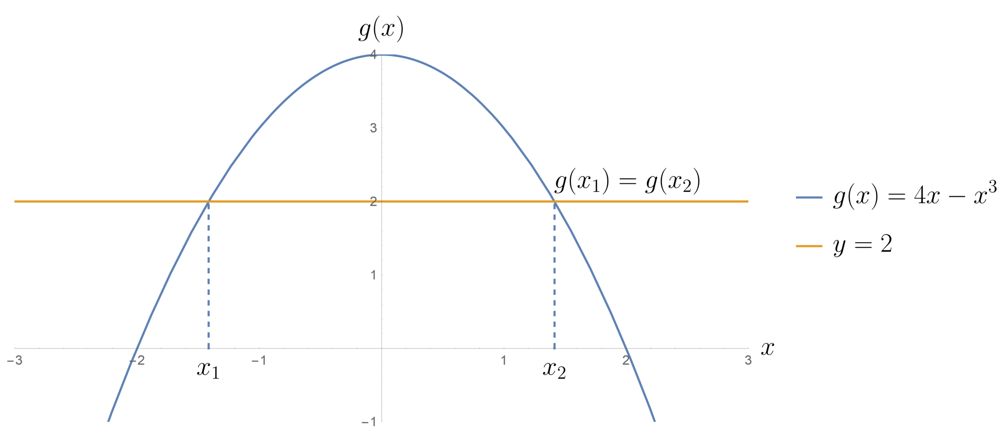


Figure 21: The function $g(x) = 4 - x^2$ fails the horizontal line test, since you can find a horizontal line that passes through it at more than one point. As shown here, this is the graphical equivalent of finding x_1 and x_2 such that $g(x_1) = g(x_2)$.

If $y = f^{-1}(x)$ then $x = f(y)$, which is just the usual equation describing the graph of the function f, $y = f(x)$, but with x and y interchanged. Thus the graph of an inverse function is obtained by switching the role of x and y, or equivalently reflecting the graph of the original function over the line $y = x$ (note that in this case the horizontal line test for the original line test corresponds to the vertical line test for the inverse function).

Whether or not a given function is invertible depends strongly on the choice of domain. For example the function $\sin : \mathbb{R} \to [-1, 1]$ is not invertible, since for example the horizontal line $y = 0.5$ passes through it infinitely many times. The function $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, however, *is* invertible:

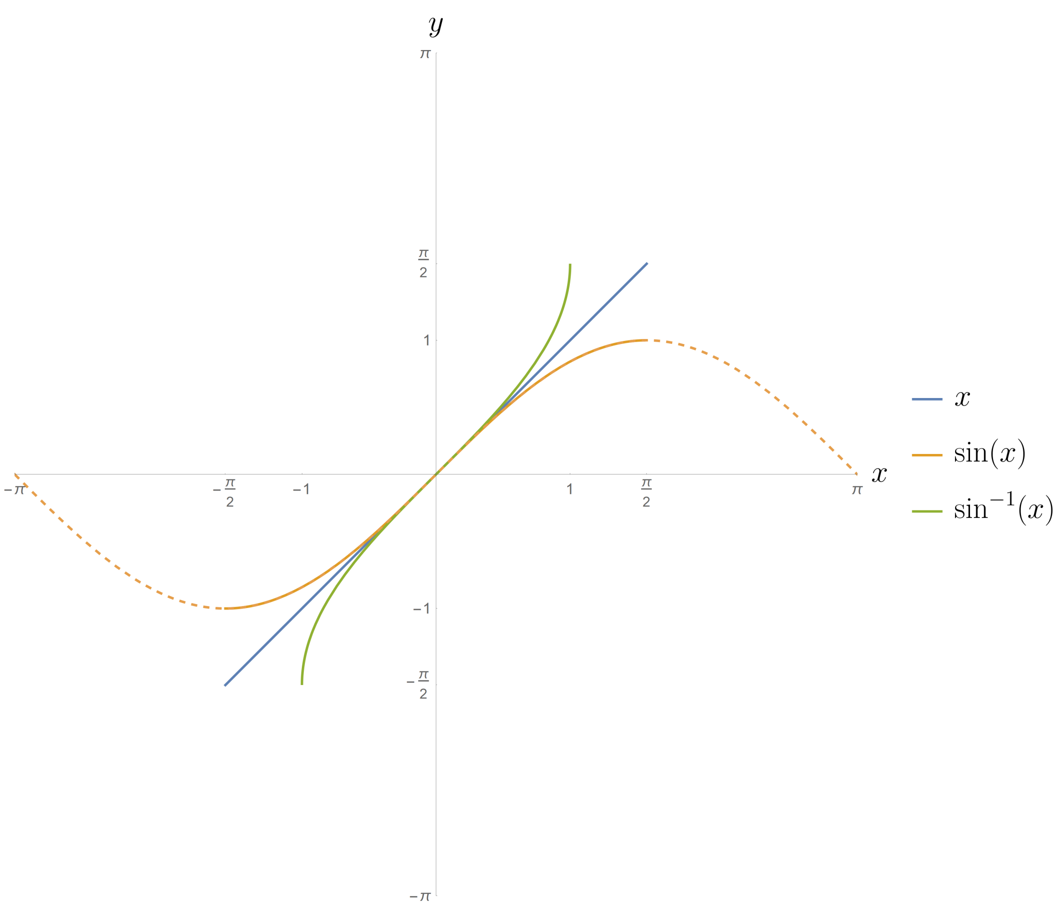


Figure 22: The function $\sin : \mathbb{R} \to [-1, 1]$ is not invertible because it fails the horizontal line test, but if we restrict to $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ (shown in orange) we can obtain an inverse $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ (shown in green) by reflecting over $y = x$ (shown in blue).

1.6 Logarithmic functions

Resources

- Khan Academy on Properties of Logarithms
- Khan Academy on Graphs of Logarithmic Functions
- Calculus with Concepts 1.8
- Single and Multi-Variable Calculus 1.4

Logarithms are the functional inverses of exponential functions. Specifically, the function $x \mapsto \log_a(x)$ is the inverse of the function $x \mapsto a^x$ – each undoes the other:

$$a^{\log_a(x)} = x \qquad \text{and} \qquad \log_a(a^x) = x \tag{1.27}$$

Another way of saying this is that $\log_a(x)$ is defined to be the unique real number y such that $a^y = x$. For example, 4^2 is 16, therefore $\log_4(16) = 2$. As noted in Section 1.5, this means that the graph of the function $x \mapsto \log_a(x)$ is the reflection of the graph of $x \mapsto a^x$ over the line $y = x$:

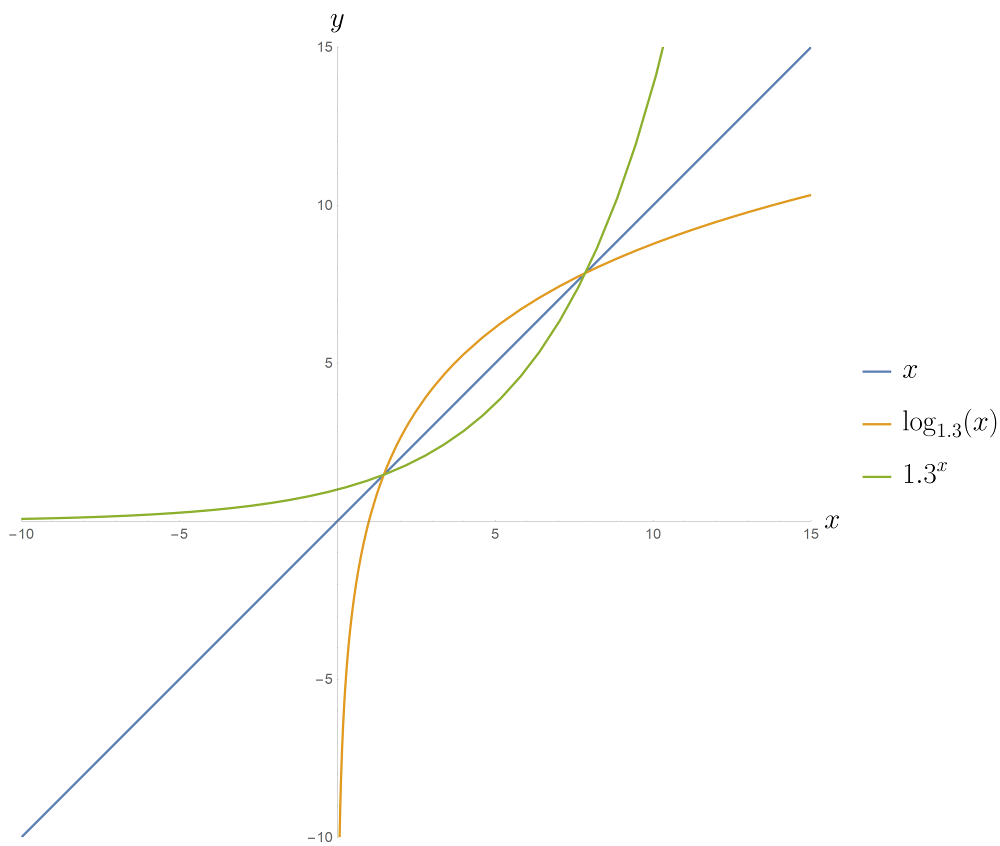


Figure 23: The graph of $\log_{1.3}(x)$ is the reflection of 1.3^x over the line $y = x$. Note that for any a the function $\log_a(x)$ "blows up" to $-\infty$ as x approaches zero from above.

The properties of logarithms can all be derived from those of the exponential, for example for

$a > 0$ and x, y real:

$$\log_a(xy) = z \iff a^z = xy \tag{1.28}$$

Meanwhile $\log_a(x) = p$ if and only if $a^p = x$ and $\log_a(y) = q$ if and only if $a^q = y$. Thus

$$\log_a(xy) = z \iff a^z = xy = a^p a^q = a^{p+q} \tag{1.29}$$

But if $a^z = a^{p+q}$ then we must have $z = p + q$, that is (substituting in for z, p, and q):

$$\log_a(xy) = \log_a(x) + \log_a(y) \tag{1.30}$$

That is to say the logarithm of a product is the sum of the logarithms (compare this to (1.24)).

Similarly, for $a, b > 0$ and x real we can compute $\log_a(b^x)$ via:

$$\log_a(b^x) = z \iff a^z = b^x \iff (a^z)^{1/x} = b \iff a^{z/x} = b \tag{1.31}$$

Thus $\log_a(b) = z/x$ and hence $z = x\log_a(b)$. Substituting in the value of z we conclude:

$$\log_a(b^x) = x\log_a(b) \tag{1.32}$$

Repeatedly applying this identity implies that a "stack" of exponents gets pulled out as a product:

$$\log_a((b^x)^y) = xy\log_b(a) \tag{1.33}$$

This should be compared with (1.25).

Finally supposing we know $\log_b(x)$ and would like to know $\log_a(x)$, how should we proceed? Well as before

$$\log_a(x) = z \iff a^z = x \tag{1.34}$$

And moreover, by the fact that the logarithm is the inverse of the exponential, $a = b^{\log_b(a)}$. Thus

$$x = a^z = (b^{\log_b(a)})^z = b^{z\log_b(a)} \tag{1.35}$$

From which (by the definition of the logarithm) we conclude $\log_b(x) = z\log_b(a) = \log_a(x)\log_b(a)$. Dividing both sides by $\log_b(a)$ we find the highly useful identity

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \tag{1.36}$$

This identity allows one to convert between logarithm bases at will, going from base b to base a one simply divides by $\log_b(a)$. A particular case of this identity to keep in the back of your head is when $x = b$:

$$\log_a(b) = \frac{1}{\log_b(a)} \tag{1.37}$$

In summary, the logarithm identities one uses most often are:

- The log of a product is the sum of logs:

$$\log_a(xy) = \log_a(x) + \log_a(y) \tag{1.38}$$

- The log of an exponential pulls out the exponent as a factor, leaving the base behind:

$$\log_a(b^x) = x\log_a(b) \tag{1.39}$$

- One can change the base of a log from b to a by dividing by $\log_b(a)$ (or multiplying by $\log_a(b)$!):

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \tag{1.40}$$

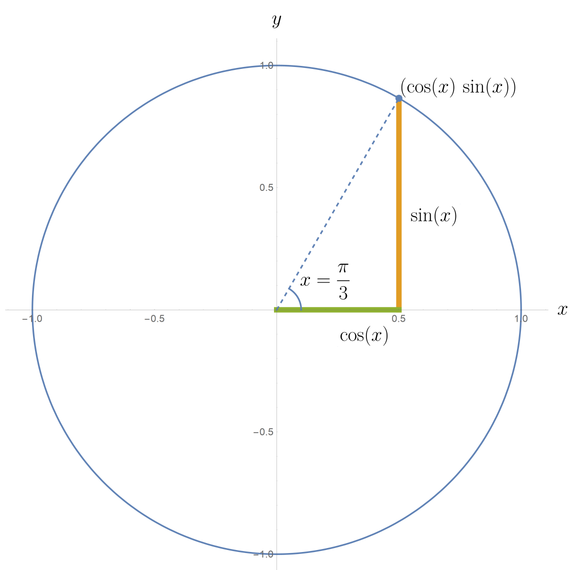
In this course the natural logarithm $\log_e(x)$ (where $e = 2.718\ldots$ is Euler's constant) will always be written as $\ln(x)$. The reason it is "ln" and not "nl" is that ln is actually short for the Latin "logarithmus naturalis." One also sometimes sees the base 10 logarithm $\log_{10}(x)$ written without its base as $\log(x)$, however this is more for historical reasons than mathematical reasons – it is unlikely that the base 10 logarithm will show up in this course.

1.7 Trigonometric functions

Resources

- Khan Academy on Trig Functions
- Calculus with Concepts 1.7
- Single and Multi-Variable Calculus 1.5

Trigonometric functions differ from linear, polynomial, exponential, and logarithmic functions in that they are difficult to motivate algebraically (at least whilst remaining among the real numbers). Instead, it is best to view them geometrically as the functions that relate *angular* quantities (e.g. arc length along a circle) to rectilinear quantities (coordinates in the xy-plane).



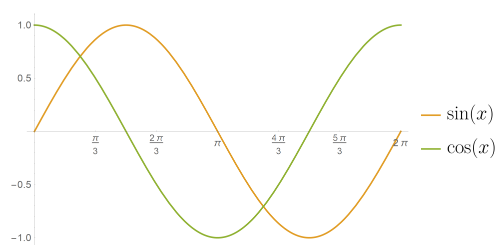


Figure 24: The standard graphical definition of $\cos(x)$ and $\sin(x)$ as the horizontal and and vertical components of a point on the circle of radius one 1 at x radians.

Note that right away this gives us our first trig identity, by the Pythagorean Theorem:

$$\cos^2(x) + \sin^2(x) = 1 \tag{1.41}$$

You should know the following identities for sin and cos:

- The sum of angles formula:

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \tag{1.42}$$
$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{1.43}$$

- The double angle formula follows from setting $x = y$ above:

$$\sin(2x) = 2\sin(x)\cos(x) \tag{1.44}$$
$$\cos(2x) = \cos^2(x) - \sin^2(x) \tag{1.45}$$

- Adding together (1.45) with (1.41) we find $2\cos^2(x) = 1 + \cos(2x)$. Taking the difference instead we find $2\sin^2(x) = 1 - \cos(2x)$. Rearranging things a bit and replacing x by $x/2$ we arrive at the half angle formulas:

$$\sin(x/2) = \pm\sqrt{\frac{1-\cos(x)}{2}} \tag{1.46}$$
$$\cos(x/2) = \pm\sqrt{\frac{1+\cos(x)}{2}} \tag{1.47}$$

  The sign $\pm$ depends on where on the unit circle $x/2$ is, for example if $x/2 \in [0, \frac{\pi}{2})$ then both are positive.

- Plugging in $y = \frac{\pi}{2}$ into the sum formulas yields the fact that $\cos(x)$ and $\sin(x)$ are horizontal translates of each other by $\frac{\pi}{2}$:

$$\cos(x) = \sin(x + \frac{\pi}{2}) \tag{1.48}$$
$$\sin(x) = \cos(x - \frac{\pi}{2}) \tag{1.49}$$

The sum of angles formula is easiest to derive with a little bit of math in the complex plane, so we'll skip the derivation for now (it is possible but tedious to derive it with plain old plane geometry).

We will also regularly use the following three trig functions:

$$\tan(x) := \frac{\sin(x)}{\cos(x)} \tag{1.50}$$
$$\cot(x) := \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)} \tag{1.51}$$
$$\sec(x) := \frac{1}{\cos(x)} \tag{1.52}$$
$$\csc(x) := \frac{1}{\sin(x)} \tag{1.53}$$

Note that unlike $\sin(x)$ and $\cos(x)$, these trig functions are not defined everywhere. For example $\tan(x)$ is undefined whenever $\cos(x) = 0$, that is whenever $x = \frac{\pi}{2} + \pi k$ for any k in $\mathbb{Z}$.

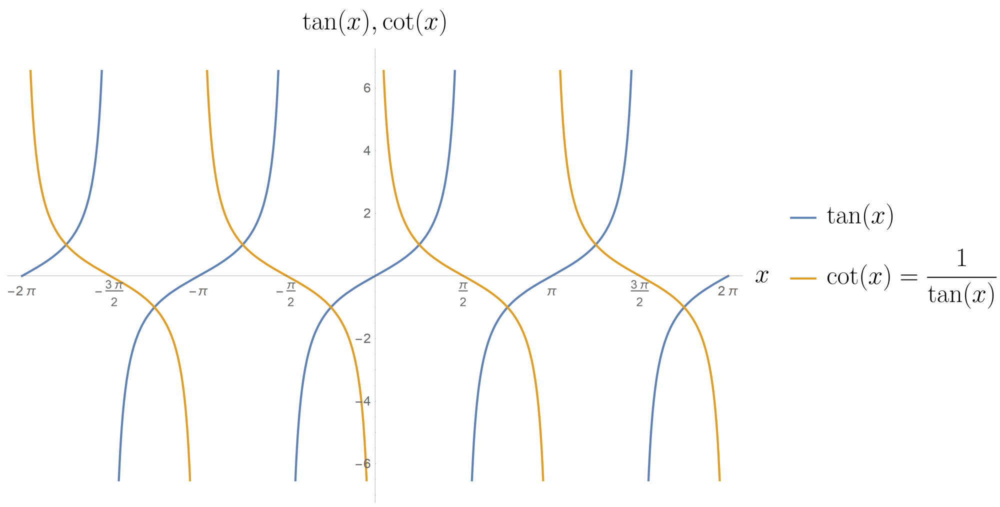


Figure 25: Graph of $\tan(x)$ and $\cot(x)$.

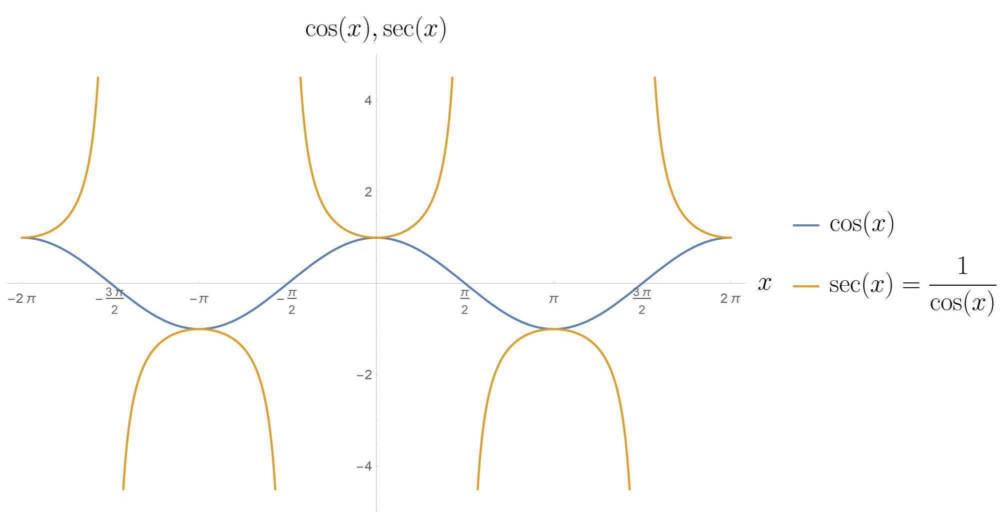


Figure 26: Graph of $\cos(x)$ and $\sec(x)$.

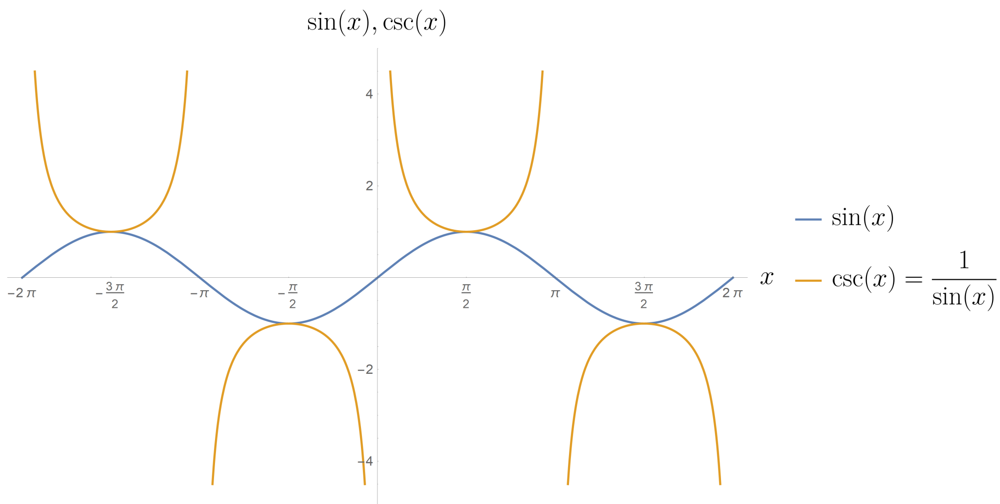


Figure 27: Graph of $\sin(x)$ and $\csc(x)$.

Some other useful trig identities that essentially follow from the definitions above are:

$$\sec^2(x) - \tan^2(x) = 1 \tag{1.54}$$
$$\csc^2(x) - \cot^2(x) = 1 \tag{1.55}$$
$$\sin(x)\tan(x) = \sec(x) - \cos(x) \tag{1.56}$$

In addition to these six trig functions, one also often has use for the *inverse* trig functions. Now, none of the trig functions mentioned above is invertible on all of $\mathbb{R}$, so that means one has to restrict the domain of the trig function to a region in which they pass the horizontal line test before taking the inverse. Or equivalently, to restrict the range of the inverse so that it passes the vertical line test:

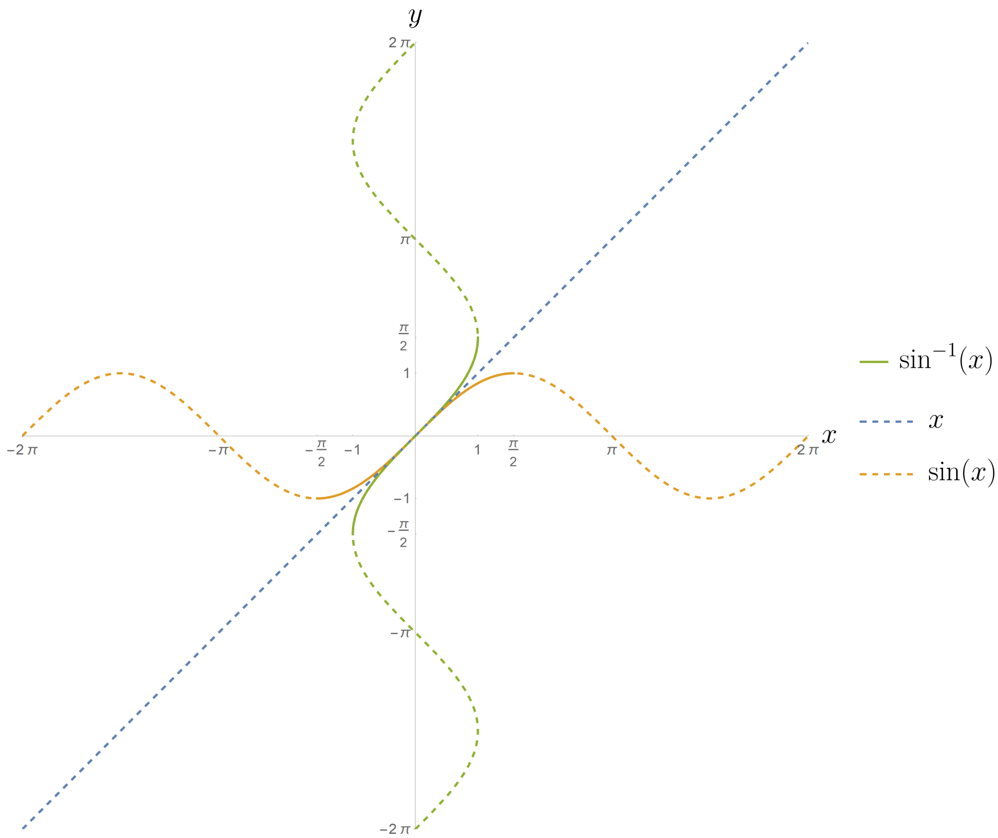


Figure 28: Graph of $\sin^{-1}(x)$.

This amounts to choosing between equivalent angles on the unit circle, for example since $\sin(\pi/2) = \sin(5\pi/2) = 1$ we could just as easily set $\sin^{-1}(1) = \pi/2$ or $\sin^{-1}(1) = 5\pi/2$. One typically makes the following choices for the ranges of the inverse functions

- $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$
- $\cos^{-1} : [-1, 1] \to [0, \pi]$
- $\tan^{-1} : (-\infty, \infty) \to (-\pi/2.\pi/2)$
- $\sec^{-1} : (-\infty, -1) \cup (1, \infty) \to (0, \pi)$
- $\csc^{-1} : (-\infty, -1) \cup (1, \infty) \to (-\pi/2, \pi/2)$

Note that in each case the domain of the inverse is the range of the original function. The graphs of the rest of the inverse trig functions are below.

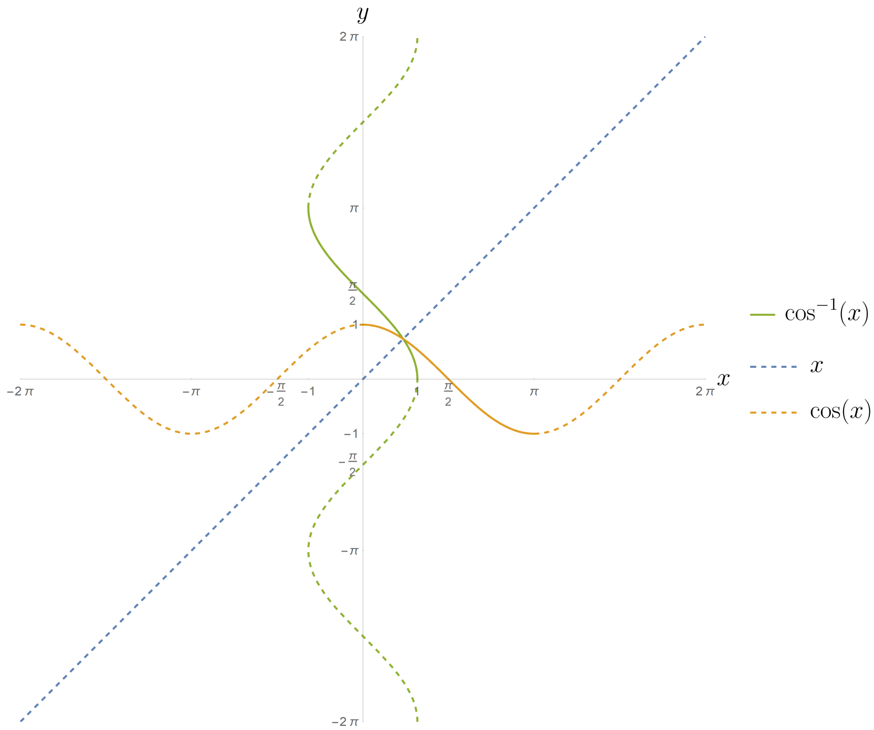


Figure 29: Graph of $\cos^{-1}(x)$.

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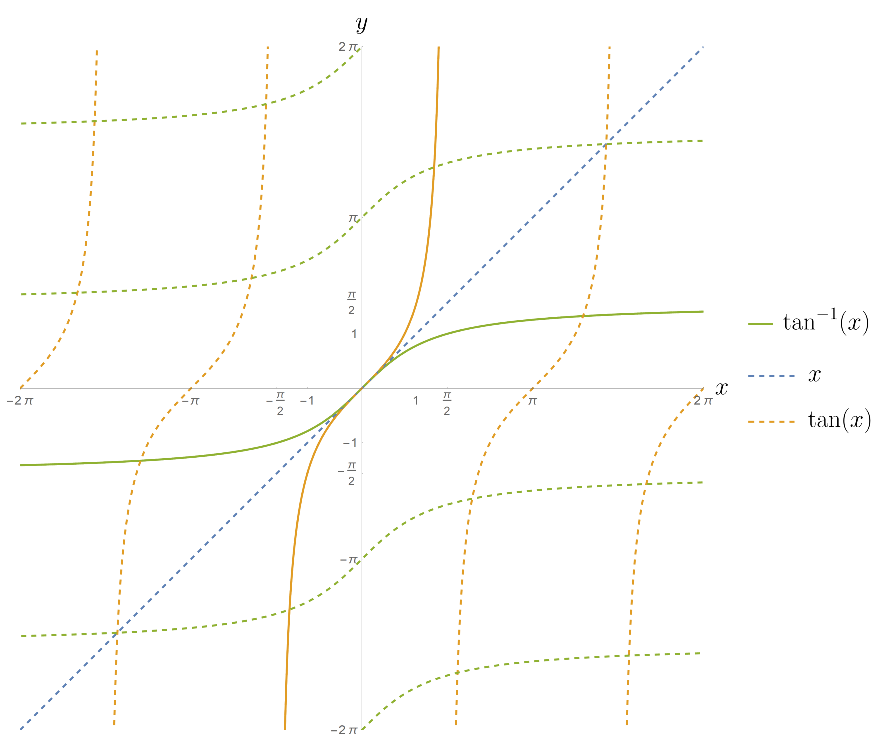

Figure 30: Graph of $\tan^{-1}(x)$.

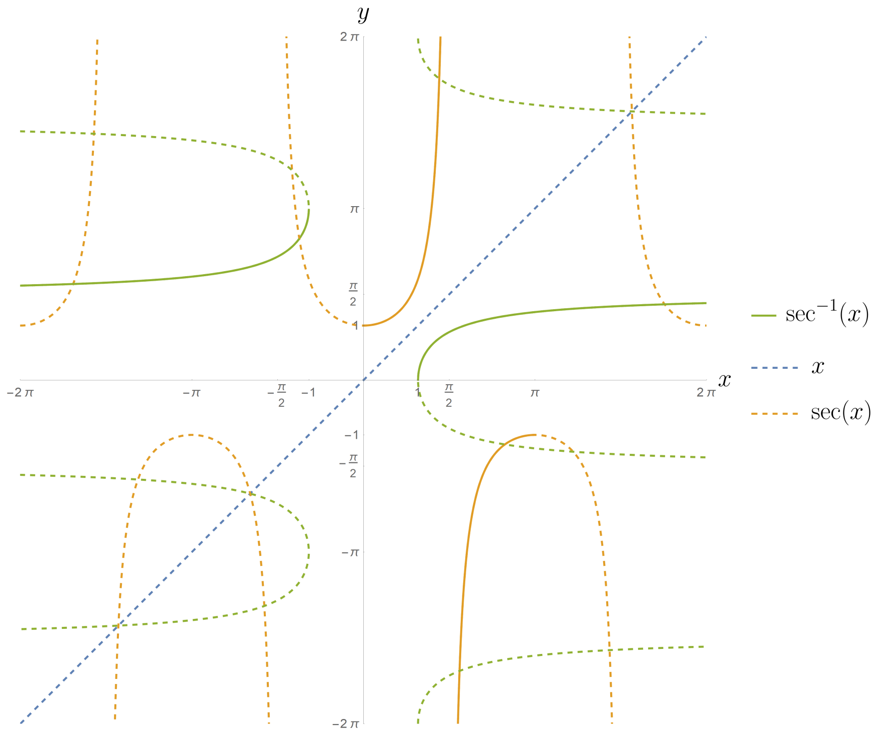

Figure 31: Graph of $\sec^{-1}(x)$.

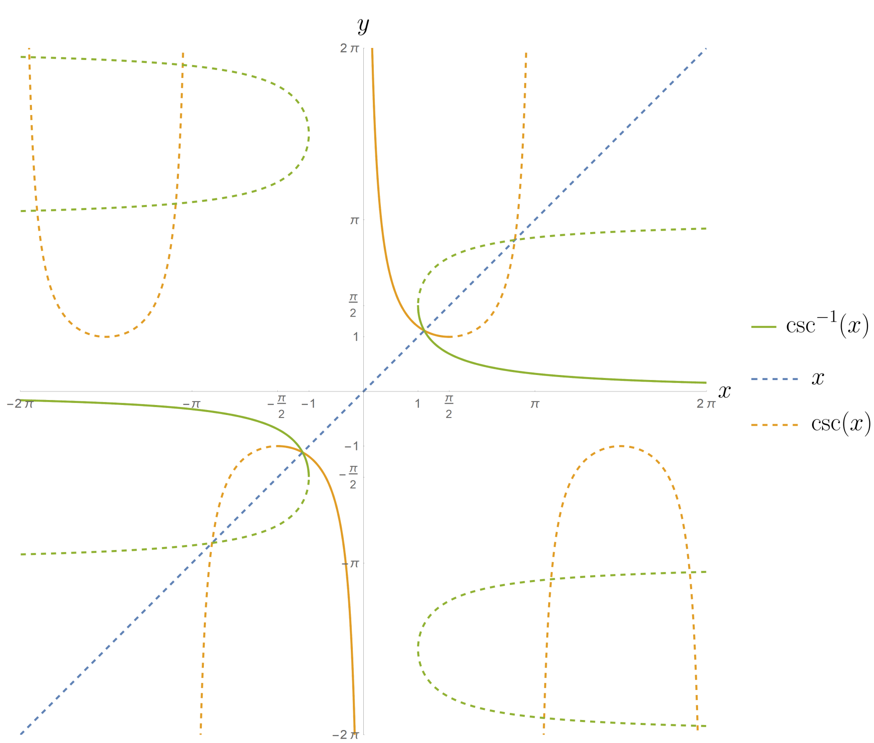

Figure 32: Graph of $\csc^{-1}(x)$.

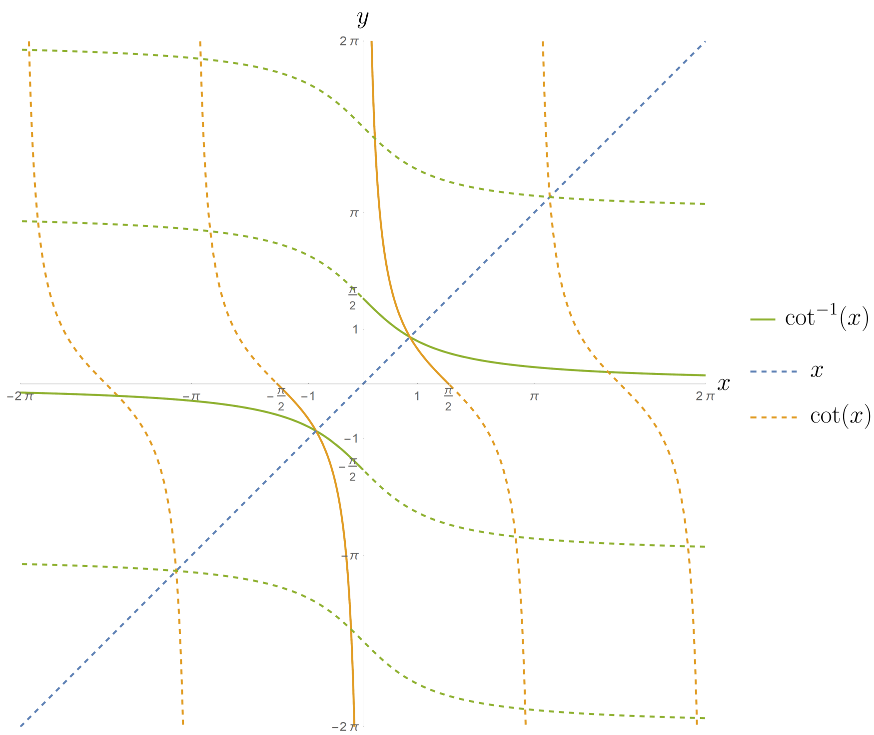

Figure 33: Graph of $\cot^{-1}(x)$.

Note that because the trig functions are not globally invertible, these trig inverses only get you back to where you started if you are in the range of the trig inverse to begin with. For example, $\sin^{-1}(\sin(\pi/4)) = \pi/4$ because $\pi/4$ is in $[-\pi/2, \pi/2]$, but $\sin^{-1}(\sin(3\pi/2)) = -\pi/2$, not $3\pi/2$.

1.8 Polynomial functions

Resources

- Khan Academy on Polynomial Functions
- Calculus with Concepts 1.3
- Single and Multi-Variable Calculus 1.6

Polynomials are one of the friendliest types of functions. A polynomial is simply a linear combination of monomials:

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \tag{1.57}$$

The numbers $c_1, \ldots, c_n$ are called the coefficients of the polynomial. An important property of polynomials is that they are defined for all $\mathbb{R}$ (and are continuous, as we shall see). As has already been noted, any polynomial with only odd powers is an odd function, whereas any polynomial with only even powers is an even function (to prove this, simply compute $f(-x)$ and factor out all of the minus signs).

Every polynomial has a *factored form* as a product of irreducible polynomials (polynomials that you can't factor any further), which over $\mathbb{R}$ looks like:

$$p(x) = c_n(x - r_1)^{m_1} \dots (x - r_k)^{m_k}((x - a_1)^2 + b_1^2)^{n_1} \dots ((x - a_l)^2 + b_l^2)^{n_k} \tag{1.58}$$

The numbers $r_1, \dots, r_k$ are the *real roots* of the polynomial, the places where the graph of the polynomial crosses the x axis. the pairs $(a_1, b_1), \dots, (a_l, b_l)$ meanwhile are also roots in the sense that $a_1 \pm b_1 i, \dots, a_l \pm b_l i$ are all roots of the polynomial (recall $i = \sqrt{-1}$ is the imaginary unit), but they are *complex roots* not real roots (unless $b = 0$). That is, if we were working over the field of complex numbers $\mathbb{C}$ we could factor the polynomial further to:

$$p(x) = c_n(x - r_1)^{m_1} \dots (x - r_k)^{m_k}(x - z_1)^{n_1}(x - \overline{z_1})^{n_1} \dots (x - z_l)^{n_l}(x - \overline{z_l})^{n_l} \tag{1.59}$$

Where $z_j = a_j + b_j i$ and $\overline{z_j} = a_j - b_j i$ is the complex conjugate of z_j. The natural numbers $m_1, \dots, m_k$ and $n_1, \dots n_l$ are the *multiplicities* of the corresponding root – you can think of this as the polynomial having multiple copies of the same root. If the multiplicity of a real root is odd, then the graph of the polynomial will pass through the x axis at the root, changing sign:

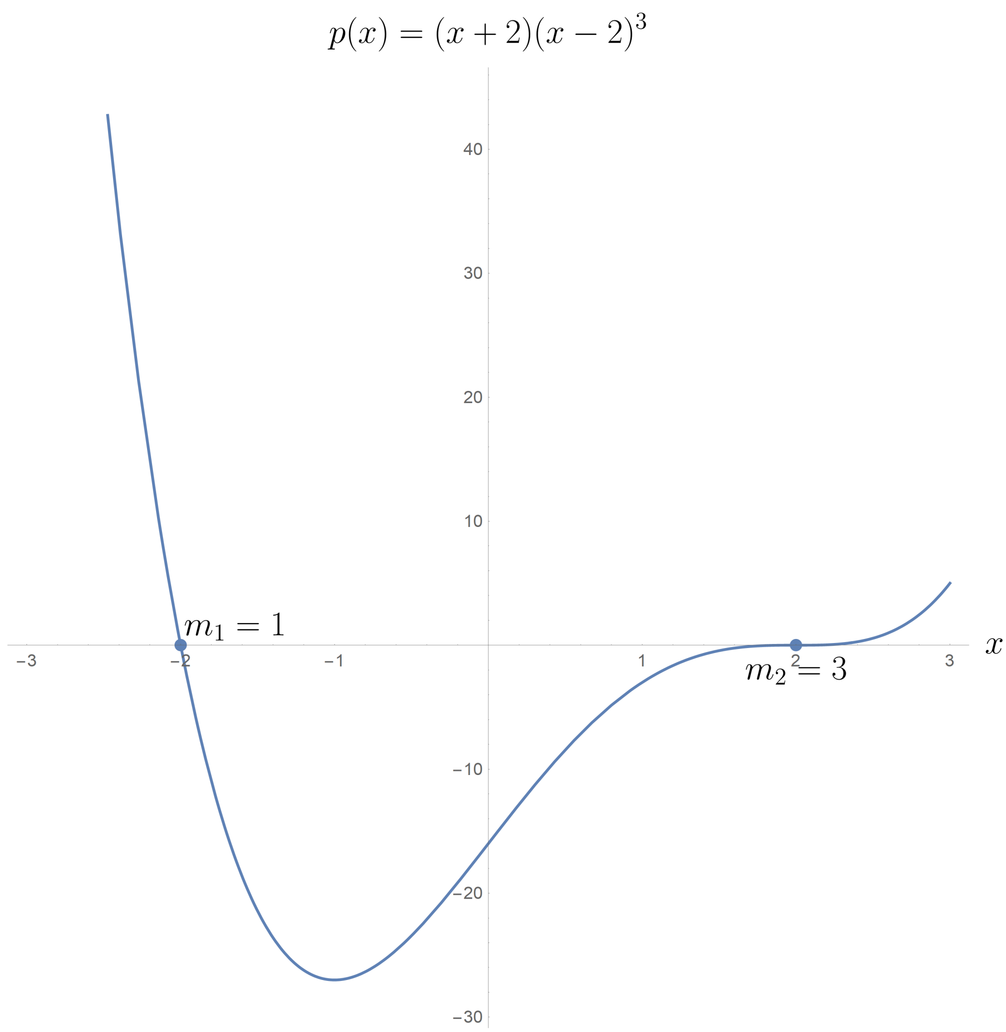

Figure 34: Note that for for the root of multiplicity 3 the graph is much flatter near the root, because the polynomial looks locally like the cubic $a(x - r)^3$ rather than the line $a(x - r)$.

Meanwhile, if the multiplicity of the root is even, then the graph of the polynomial will still go to zero at the root, but without changing sign, it will “bounce off” the x-axis:

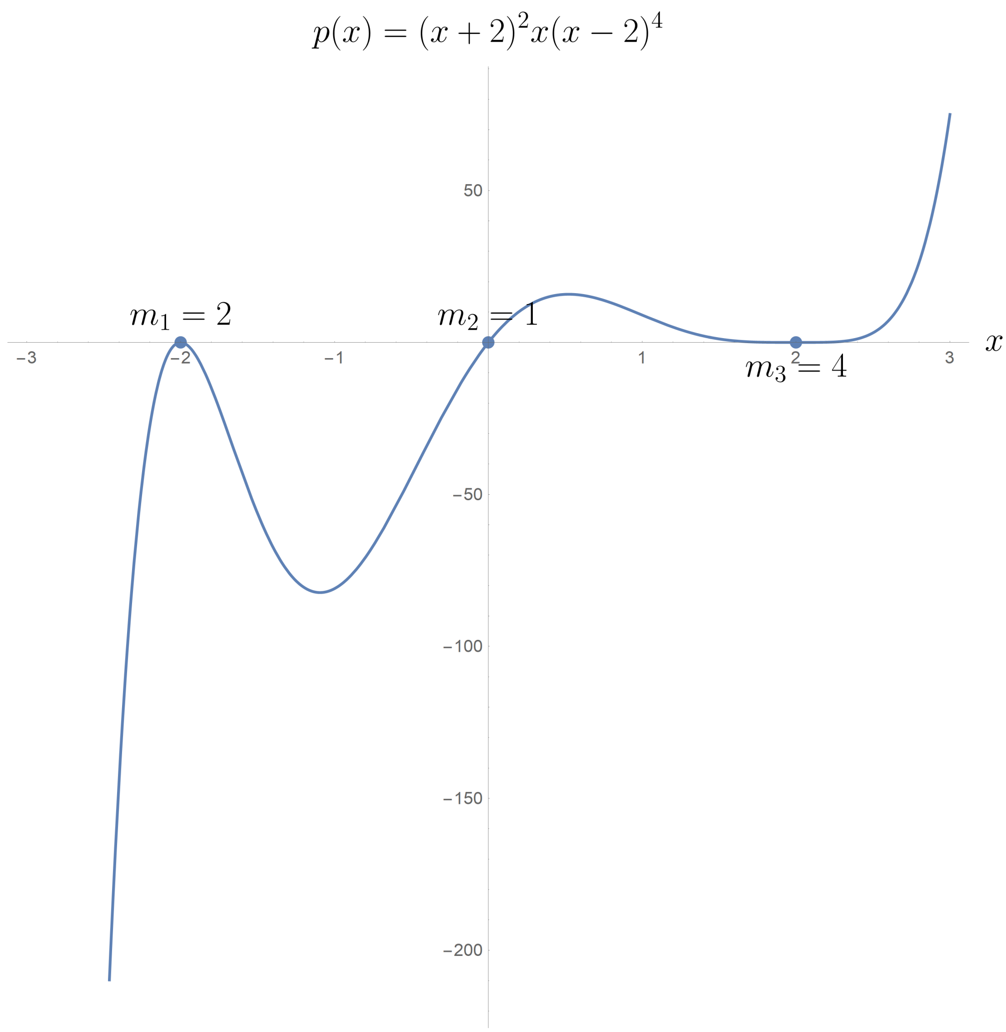

Figure 35: Note that for the root of multiplicity 4 the graph is much flatter near the root than it is for the root of multiplicity 2, because the polynomial looks locally like $a(x-r)^4$ instead of $a(x-r)^2$.

Geometrically, complex roots tell you where the graph of the polynomial "changes direction" and thereby avoids going through the x axis:

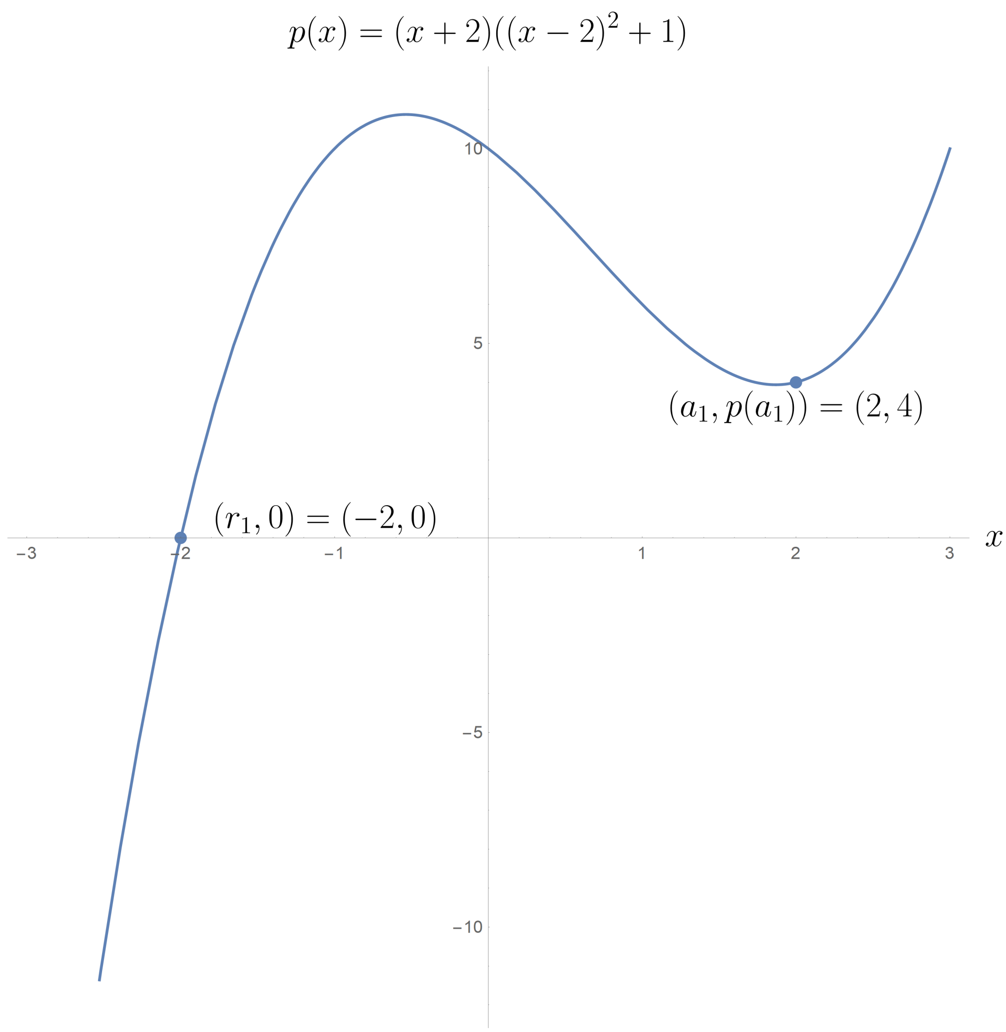

Figure 36: Note that while $a_1 = 2$ tells you approximately where the polynomial will turn (since close to $x = a_1$ the polynomial looks like $p(x) = (\text{const})((x - a_1)^2 + b_1^2)$), it does not tell you *exactly* where the local minimum is for the polynomial (in this case it is slightly to the left of $x = a_1$).

It as assumed you know how to find the roots of a quadratic polynomial, or complete the square if no real roots exist. There are various tricks to factoring a polynomial (putting it into the factored form (1.58)) when $n = 3$ or $n = 4$, although it must be noted that in the general case it is extremely difficult. One trick is to add and subtract something clever, for example if $p(x) = x^3 - 2x + 1$ then:

$$\begin{aligned} p(x) &= x^3 - 2x + 1 \\ &= x^3 - x^2 + x^2 - 2x + 1 \\ &= x^2(x-1) + (x-1)^2 \\ &= (x^2 + x - 1)(x - 1) \\ &= (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2})(x-1) \end{aligned} \tag{1.60}$$

In this course it will be assumed that you know how to factor such polynomials where possible, and that you know how to do polynomial long division to factor a polynomial that you have found a root for.

To sketch the graph of a polynomial one essentially needs to know three things: the polynomial's roots, their multiplicities, and what happens to $p(x)$ as $x \to -\infty$ (or $+\infty$). The reason one doesn't need to check the sign in between the roots is that if one works left to right, the multiplicity of the root tells you what it is: the sign changes when the multiplicity of a root is odd and it stays the same when it is even. To figure out what happens as $x \to -\infty$, just look at the leading term in the polynomial (as everything else will be negligible for $|x|$ sufficiently large).

Example 1.14. Sketch the graph of $p(x) = (1-x)x^2(x-2)^2$.

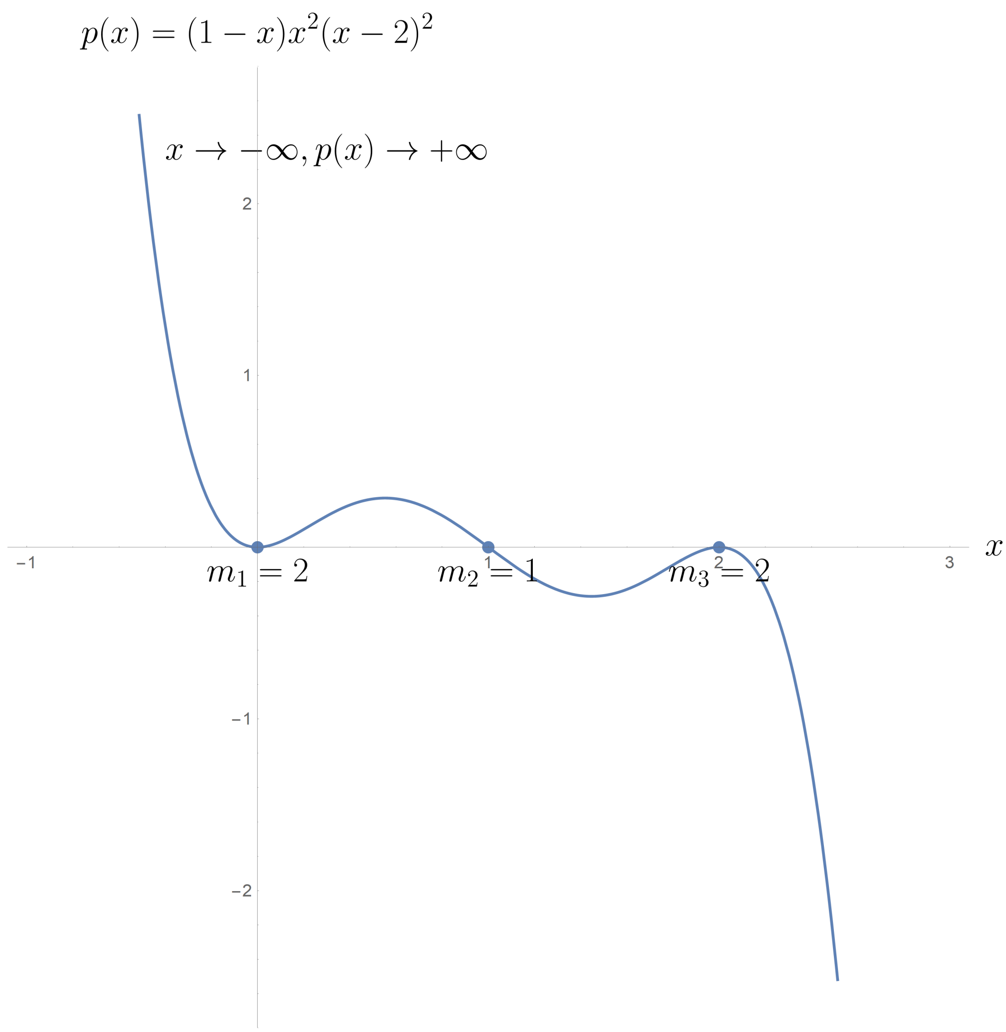

Figure 37: The leading term here (multiplying together the leading terms of each factor) is $-x^5$, which goes to $+\infty$ as $x \to -\infty$. We encounter a root of multiplicity 2 at $x = 0$, so the sign doesn't change and the graph just "bounces off" the x axis, then a root of multiplicity 1 at $x = 1$, so the sign changes and the graph passes through the x axis, and finally a root of multiplicity 2 at $x = 2$, so the sign stays negative and the graph bounces off the x axis in the negative direction.

1.9 Rational functions and asymptotes

Resources

- Khan Academy on Rational Functions
- Calculus with Concepts 1.3
- Single and Multi-Variable Calculus 1.6

Rational functions are so called not because they are clever, but because they are ratios, specifically of polynomials:

$$r(x) = \frac{p(x)}{q(x)} \tag{1.61}$$

The largest possible domain of such a function is of $\mathbb{R} \setminus \{r_1, \ldots, r_k\}$ where $r_1, \ldots, r_k$ are the roots of the polynomial q.

Like polynomials, rational functions are characterized by their roots and their asymptotic or long term behavior as $x \to \pm\infty$, however unlike polynomials they are also characterized by *discontinuities*. We will return later to a more rigorous definition of continuity, but for now just note that in this class we will encounter three types of discontinuities:

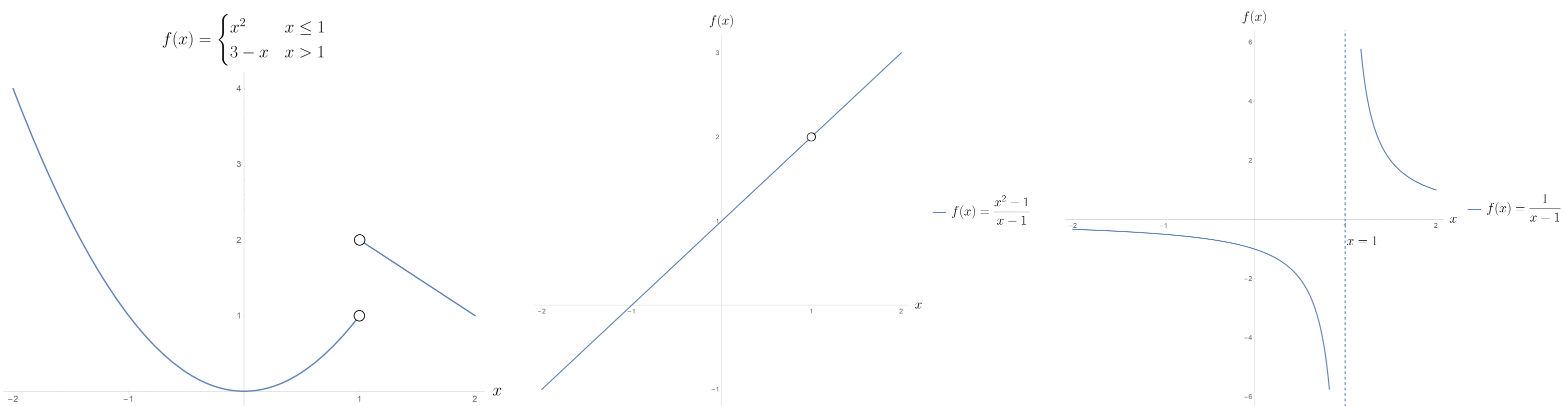

Figure 38: From left to right, these examples have 1) a jump discontinuity, 2) a hole discontinuity or removable discontinuity, and 3) a divergent discontinuity.

Rational functions only exhibit hole discontinuities and divergent discontinuities – they never jump:

- A rational function $r(x) = p(x)/q(x)$ has a *divergent discontinuity* or *vertical asymptote* at $x = x_0$ if $x = x_0$ is a root of q of multiplicity m and $x = x_0$ is either not a root of p or it is a root of p of multiplicity smaller than m.

- A rational function has a *hole* or a *removable discontinuity* at $x = x_0$ if x_0 is a root of p of multiplicity m and a root of q of multiplicity n with $m \geq n$. The hole can be removed by simplifying the rational expression, leaving behind a root of multiplicity $m - n$ if $m > n$.

- A rational function has a *root* at $x = x_0$ if $p(x_0) = 0$ and $q(x_0) \neq 0$.

In addition to roots, holes, and vertical asymptotes, it is also useful to analyze the long term behavior of rational functions as $x \to \infty$ and $x \to -\infty$. As $|x|$ becomes very large, only the leading order terms of the numerator and denominator of the rational function are significant:

$$r(x) = \frac{a_n x^n + \ldots + a_0}{b_m x^m + \ldots + b_0} \sim \frac{a_n}{b_m} x^{n-m} \qquad \text{as } |x| \to \infty \tag{1.62}$$

Note that there are three qualitative types of long term behavior:

- Case 1: $n > m$. If $n > m$ then $r(x) \sim (a_n/b_m)x^{n-m}$ diverges as $|x| \to \infty$. Whether $(a_n/b_m)x^{n-m}$ blows up to $+\infty$ as $x \to +\infty$, to $-\infty$ as $x \to +\infty$, and to $+\infty$ as $x \to -\infty$, or $-\infty$ as $x \to -\infty$ depends on the signs of a_n and b_m and on whether $n - m$ is even or odd.
- Case 2: $n = m$. If $n = m$ then $r(x) \sim a_n/b_m$ approaches a constant as $|x| \to \infty$. In this case the graph has a horizontal asymptote at $y = a_n/b_m$.
- Case 3: $n < m$. If $n < m$ then $r(x) \sim (a_n/b_m)x^{n-m}$ goes to zero as $|x| \to \infty$, and the graph has a horizontal asymptote at $y = 0$.

Example 1.15. Let $r(x) = \frac{x^2(x-2)^3}{(x-2)(x-1)}$. This function has a root of multiplicity 2 at $x = 0$, a hole at $x = 2$ that reduces to a root of multiplicity 2 once it is removed, and a vertical asymptote at $x = 1$. Moreover $p(x) = x^2(x-2)^3$ with leading order term x^5 is a quintic, whereas the denominator $q(x) = (x-2)(x-1)$ has leading order term x^2, so asymptotically $r(x)$ behaves like $x^{5-2} = x^3$. If we plot all of this information we get something like:

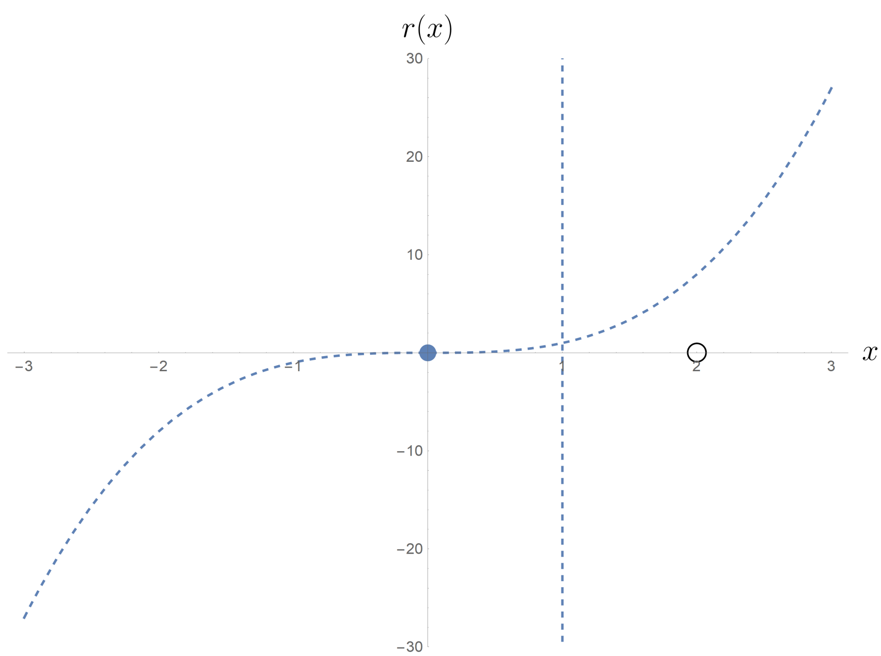

Figure 39: This plot captures the asymptotic behavior of $r(x)$ as being $\sim x^3$, the vertical asymptote at $x = 1$, the root at $x = 0$ and the hole at $x = 2$.

In order to actually plot $r(x)$, we should think about what it is doing near its roots, holes, and asymptotes. When $x \to 0^-$ (x approaches 0 from below) the numerator of $r(x)$ is negative and the denominator is positive, so $r(x)$ is negative. Meanwhile 0 is a root of multiplicity 2, so the graph should bounce off the x axis in the negative direction. As $x \to 1^-$, the numerator of $r(x)$ is negative while the denominator is positive, so $r(x)$ blows up to $-\infty$. Meanwhile, as $x \to 1^+$ the numerator of $r(x)$ is negative and the denominator is also negative, so $r(x)$ blows up to $+\infty$. Finally, approaching the hole at $x = 2$ from below the numerator is negative and the denominator is also negative, so $r(x)$ is positive. But the hole leaves behind a root of multiplicity two, so $r(x)$ bounces off of the x axis in the positive direction. Taking all of this information into account we can start to sketch the graph of $r(x)$ as:

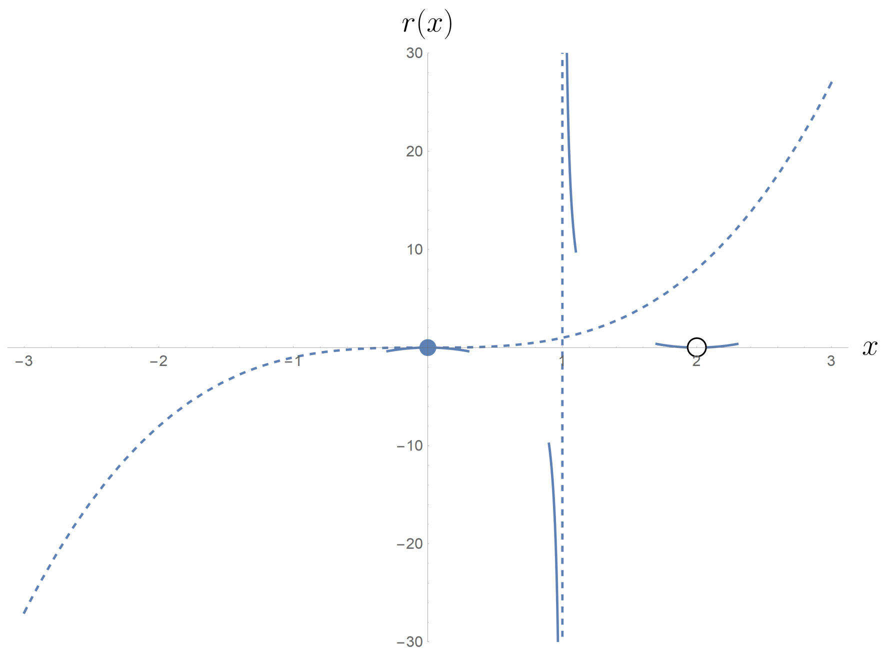

Figure 40: This plot starts to fill in the behavior of $r(x)$ near its vertical asymptote and near its roots and holes.

At which point we can just kind of "connect the dots" to obtain a reasonable sketch of the graph for $r(x)$:

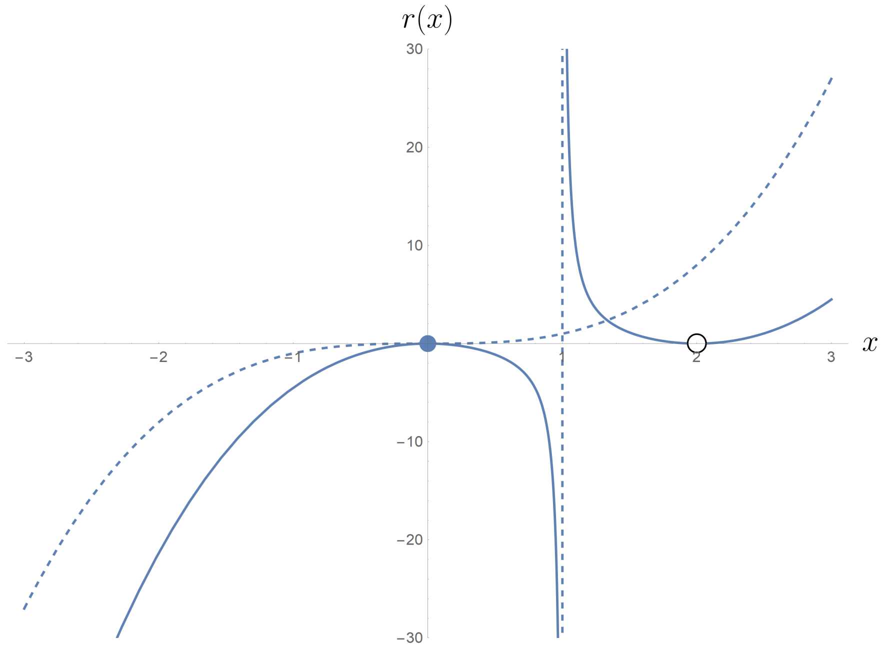

Figure 41: The graph of $r(x)$ for $-3 \leq x \leq 3$.

But hang on! Didn't we say it should behave like x^3 as $|x|$ becomes large? Well yes, but 3 is not particularly large. Let's plot the same function in the interval $[-20, 20]$:

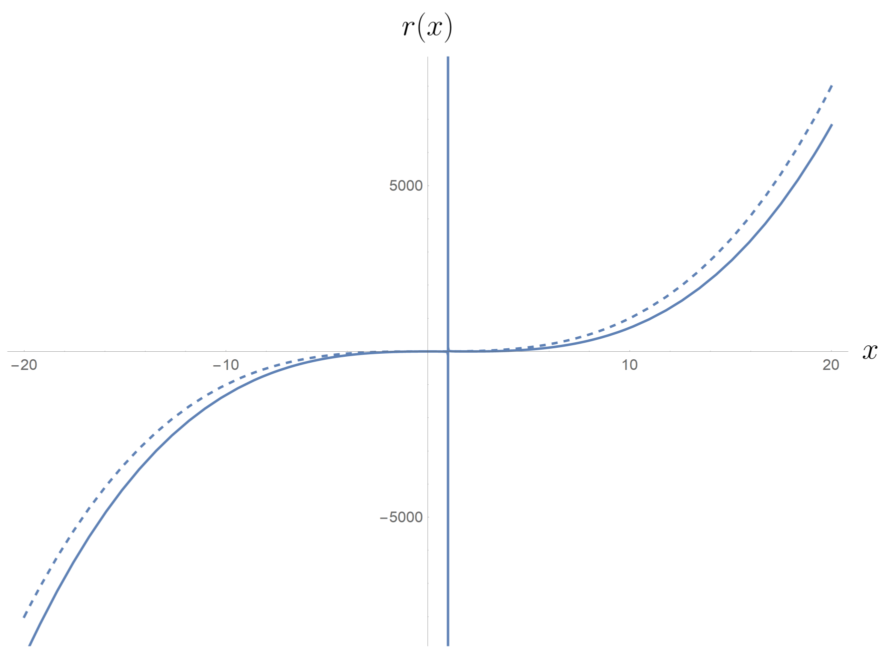

Figure 42: The graph of $r(x)$ for $-20 \leq x \leq 20$.

You can see that as promised it starts to look pretty close to x^3 as $|x| \to \infty$.

Let's do another example, this time one with a horizontal asymptote in addition to a vertical asymptote.

Example 1.16. Let $r(x) = \frac{2x(x+1)}{(x-1)^2}$. To leading order in the numerator in the denominator $r(x) \sim 2x^2/x^2 = 2$. Thus we have a horizontal asymptote $y = 2$. We also have a vertical asymptote at $x = 1$, and two roots each of multiplicity 1 at -1 and 1. We work from left to right. Just to the left of -1, the numerator and the denominator of $r(x)$ are both positive. Since -1 is a root of multiplicity 1, the function must switch signs at -1, and then again at 0. As $x \to 1^-$, both the numerator and the denominator of $r(x)$ are positive, and so the function blows up to $+\infty$. Similarly for $x \to 1^+$. With all this information we may then sketch the graph of $r(x)$:

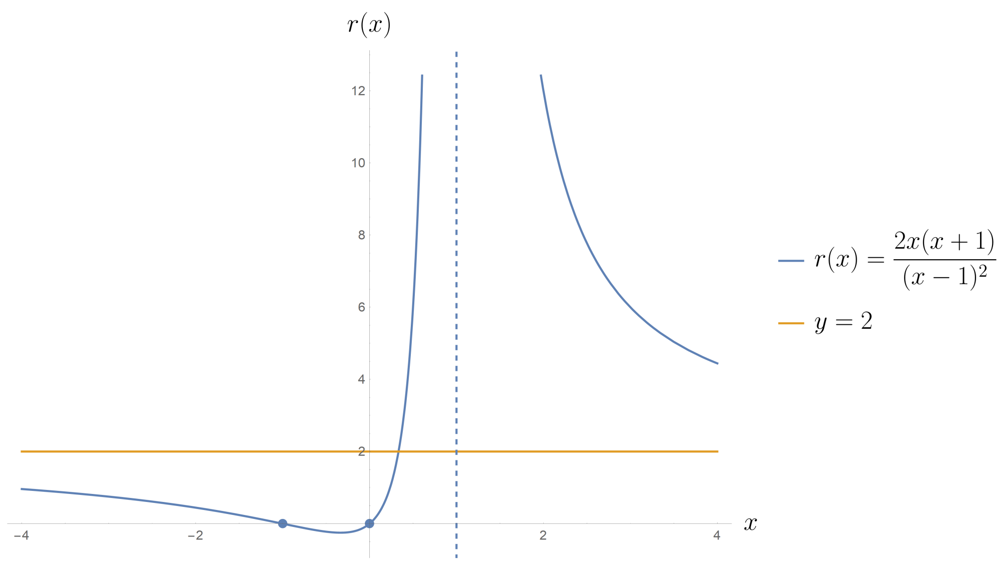

Figure 43: Graph of $r(x)$ when $-4 \le x \le 4$.

2 Limits, Continuity, and Derivatives

2.1 Limits, Epsilon-Delta Calculus

Resources

- Khan Academy on Limits and Continuity
- 3Blue1Brown on Limits
- 3Blue1Brown on Epsilon-Delta Calculus and L'Hôpital's rule
- Calculus with Concepts 2.2, 2.3, 2.4
- Single and Multi-Variable Calculus 1.8

The intuitive notion of a limit is just that, pretty intuitive:

$$\lim_{x \to c} f(x) = L \iff f(x) \text{ “approaches” } L \text{ as } x \text{ “approaches” } c \tag{2.1}$$

The tricky part here is the word approaches, but hopefully what it means will become clear after a few examples before we rigorously define the limit.

Example 2.1. What is $\lim_{x \to 2} x^2$? Well we might guess 4 since $2^2 = 4$, but the limit doesn't care about the value of the function at c, just what happens as we get closer and closer to it. So let's look at x^2 for a few numbers that get closer and closer to 2:

x	x^2
1.9	3.61
1.99	3.9601
1.99	3.996001
$\vdots$	$\vdots$

Looks pretty good for the limit being 4, so let's also check what x^2 does for values of x getting closer and closer to 2 from above:

x	x^2
2.1	4.41
2.01	4.0401
2.001	4.004001
$\vdots$	$\vdots$

We haven't proved it, because we haven't even really said what a limit is yet, but with our notion of limit we seem to have quite a lot of evidence that $\lim_{x \to 2} x^2 = 4$ (and this is in fact the case).

Let's do an example where the $\lim_{x \to c} f(x)$ turns out to be different from $f(c)$.

Example 2.2. Let $g(x) = \frac{x-2}{x^2-4}$ and consider $\lim_{x \to 2} g(x)$. The function $g(x)$ is a rational function with a hole at $x = 2$ and a vertical asymptote at $x = -2$. Away from $x = 2$ we have

$$g(x) = \frac{1}{x+2} \qquad \text{for } x \neq 2 \tag{2.2}$$

Thus we might suspect that $\lim_{x\to 2} g(x) = \frac{1}{2+2} = \frac{1}{4}$. We would be right! But let's look at a few values of $g(x)$ as x gets closer and closer to 2 anyway:

x	$g(x) = \frac{1}{x+2}$
1.9	$\frac{1}{3.9} = 0.256\ldots$
1.99	$\frac{1}{3.99} = 0.2506\ldots$
$\vdots$	$\vdots$
2.1	$\frac{1}{4.1} = 0.244\ldots$
2.01	$\frac{1}{4.01} = 0.249\ldots$
$\vdots$	$\vdots$

It would appear that indeed $\lim_{x\to 2} g(x) = \frac{1}{4}$ even though $g(2)$ is not defined. It's worth repeating, *the limit doesn't care about the value of the function at the limit point*. Indeed, even if we defined $g(x)$ arbitrarily to take some other value at 2, the limit would remain unchanged. Let $\tilde{g} : \mathbb{R} \to \mathbb{R}$ be defined via:

$$\tilde{g}(x) = \begin{cases} g(x) & x \neq 2 \\ 1 & x = 2 \end{cases} \tag{2.3}$$

The function $\tilde{g}$ has graph:

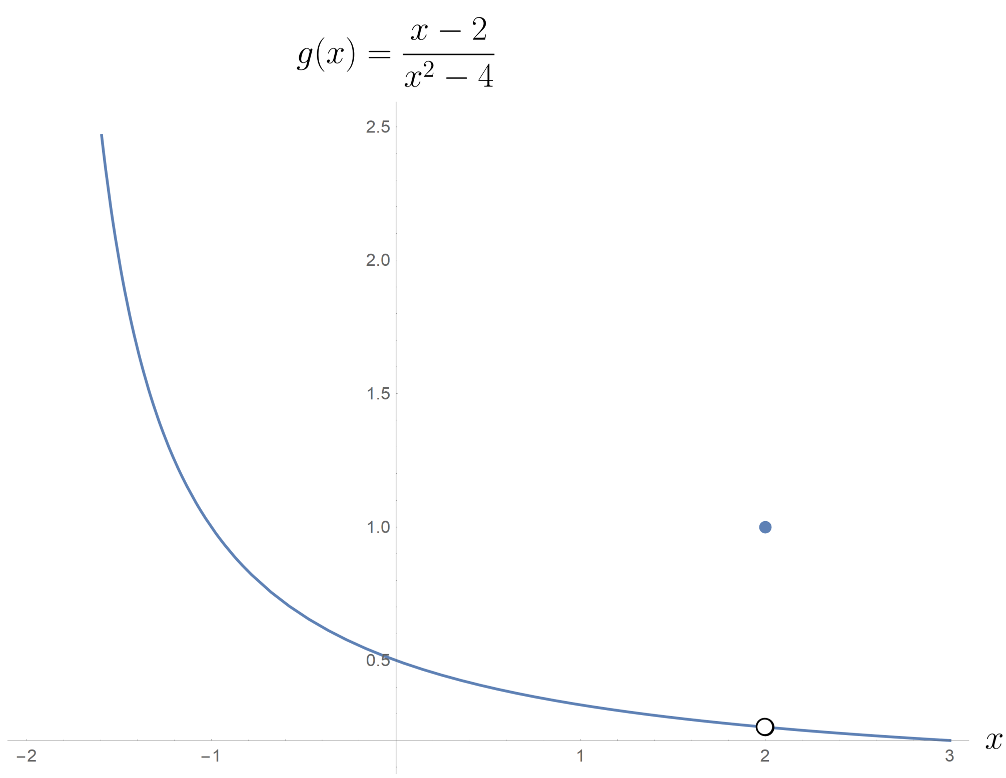

Figure 44: The function $\tilde{g}$ is discontinuous at $x = 2$.

Here $\lim_{x\to 2} \tilde{g}(x)$ is still equal to $\frac{1}{4}$ even though $\tilde{g}(2) = 1$ ($\tilde{g}$ agrees with g everywhere except for 2 and the limit doesn't care about what happens at 2).

Before going any further, we should probably ask the question "why are limits useful?" Well, they turn out to be useful for understanding lots of phenomena "in the long term" or "near a critical point," but in this class their most powerful application will be in defining the derivative. Not to get too ahead of ourselves, but we will be interested in the following quantity:

$$f'(x) \text{ (aka } \tfrac{df}{dx}) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2.4}$$

That is, $f'(x)$ is defined to be the limit of the slopes of the secant lines from x to $x+h$ as h approaches 0. But back to limits for the moment, we need a formal, rigorous definition of a limit. The word "approach" is doing a lot of work here, and we need to state its meaning with mathematical precision. Why? Well

- This is math! We need to define what we're working with.

- If we want to understand useful properties of limits, we need a formal definition in order to prove those properties.

- The formal definition of the limit is actually the first glimpse most students get of the wonderful field of real analysis. The relationship between calculus and real analysis is sort of analogous to the relationship between anatomy and microbiology, or possibly even organic chemistry. Real analysis is the "underpinning" of everything we will do in this class.

Definition. We say that $\lim_{x \to c} f(x) = L$ if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ we have $|f(x) - L| < \epsilon$.

If this definition makes little sense, don't panic, you're in good company. Hopefully by the end of this section we will have shed some light on why it is the correct definition of the limit. What this definition says is that a limit is really a game: I hand you an $\epsilon > 0$ that I want to squeeze $f(x)$ close to L with, that is I want $f(x) \in (L - \epsilon, L + \epsilon)$. Your goal is to produce a $\delta > 0$ to make this happen whenever x is within δ of c, that is $x \in (c - \delta, c + \delta)$. If you can always win at this game, no matter how tiny an ϵ I hand you (but still greater than zero!), then we say $\lim_{x \to c} f(x) = L$.

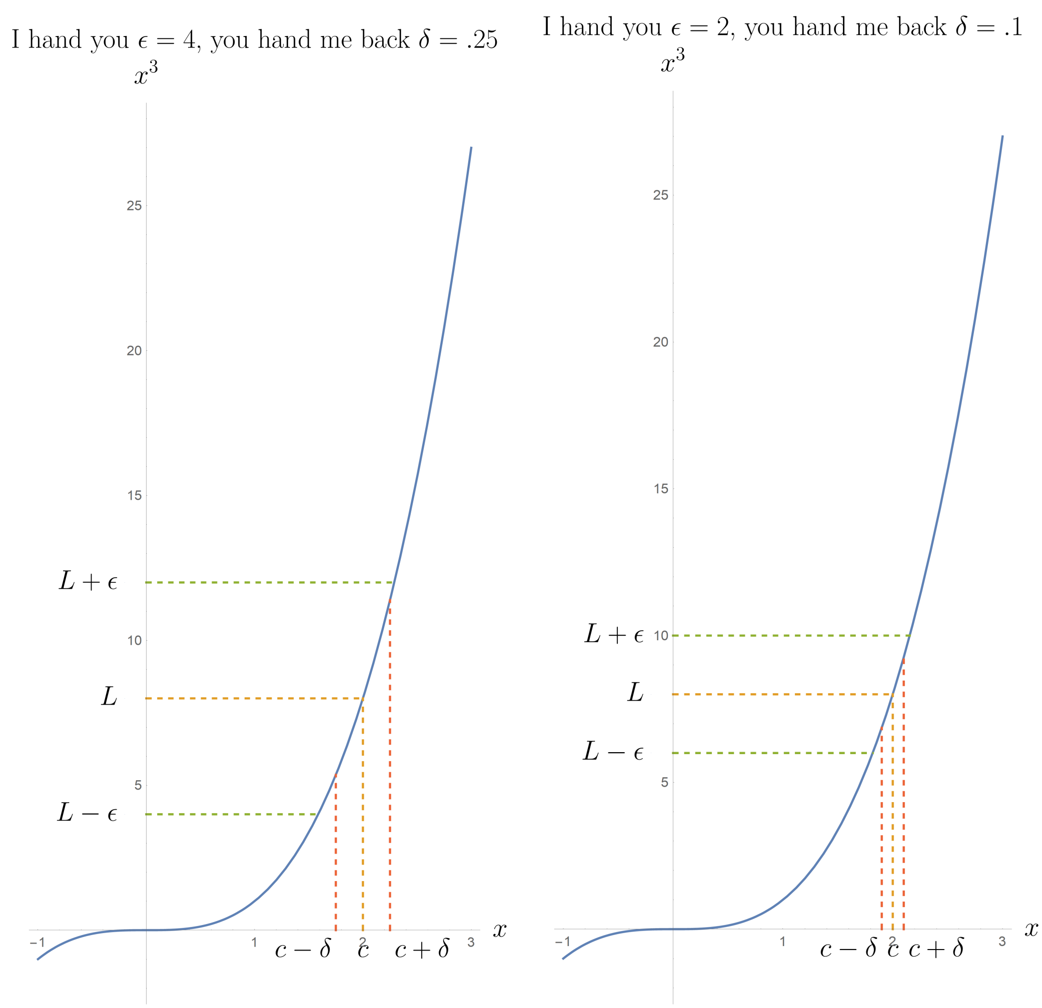

Figure 45: The ϵ-game for $\lim_{x\to 2} x^3 = 8$. If you can win this game no matter how small an ϵ you are given, then the limit is correct.

Lets do a few examples of proving limits using this $\epsilon - \delta$ definition.

Example 2.3. Prove that $\lim_{x\to 2} 5x = 10$. Let's begin. I hand you an $\epsilon > 0$, and you have to hand me back a $\delta > 0$ so that whenever $|x-2| < \delta$ we will get $|5x-10| < \epsilon$. Let's see if we can upper bound $|5x-10|$ in terms of δ:

$$\begin{aligned} |5x-10| &= 5|x-2| \\ &< 5\delta \end{aligned} \tag{2.5}$$

Where to get the second line we used the fact that we can assume $|x-2| < \delta$! We want this upper bound to be ϵ, and we can pick any $\delta > 0$, so pick $\delta = \frac{\epsilon}{5}$ to get

$$|5x-10| < 5\frac{\epsilon}{5} = \epsilon \tag{2.6}$$

Thus, triumphant, you hand me back $\delta = \frac{\epsilon}{5}$ with the guarantee that whenever $|x-2| < \delta$ I will get $|5x-10| < \epsilon$. But wait I say! That was just one ϵ, I have infinitely many more for you to produce a δ for! You shake your head, "ϵ was arbitrary, hand me ϵ and I will hand you back $\delta = \epsilon/5$ every time, I will always win the game." You have proved that $\lim_{x\to 2} 5x = 10$!

Often times we will use the shorthand $\forall$ to mean "For All", $\exists$ to mean "There Exists", : to mean "Such That" and $\implies$ to mean "Implies." With these notations available the definition of the limit becomes the rather frightening set of hieroglyphics:

$$\lim_{x\to c} f(x) = L \iff \forall \epsilon > 0 \exists \delta > 0 : |x-c| < \delta \implies |f(x) - L| < \epsilon \tag{2.7}$$

When you see such expressions try to read them out loud.

Example 2.4. Prove $\lim_{x\to 2} x^2 = 4$. We motivated this limit earlier by simply looking at x^2 for values of x closer and closer to 2, but now we will prove it. Fix $\epsilon > 0$, then

$$\begin{aligned} |x^2 - 4| &= |x-2||x+2| \\ &< \delta |x+2| \end{aligned} \tag{2.8}$$

It might seem like we have a problem here, because we need to upper bound $|x+2|$ which is in general an unbounded function, but recall that we have $|x-2| < \delta$ and we may set δ however we wish so long as it is greater than zero. Thus we will restrict ourselves to $\delta < 1$ in which case $|x-2| < \delta < 1$ implies that $x \in (1,3)$, which implies that $|x+2| \in (3,5)$ and in particular $|x+2| < 5$. Thus if $\delta \leq 1$ then:

$$|x^2 - 4| < 5\delta \tag{2.9}$$

We need this upper bound to be ϵ, while still maintaining $\delta \leq 1$, so set $\delta = \min(1, \frac{\epsilon}{5})$ and we conclude

$$|x^2 - 4| < \epsilon \tag{2.10}$$

Thus, you hand me ϵ and I hand you back $\delta = \min(1, \frac{\epsilon}{5})$, which we have just shown guarantees that $|x-2| < \delta$ implies $|x^2 - 4| < \epsilon$, so $\lim_{x\to 2} x^2 = 4$.

Theorem 2.1. *Let f and g be functions that are defined on an interval around c such that $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.*

(a) $\lim_{x\to c} bf(x) = b \lim_{x\to c} f(x)$.

(b) $\lim_{x\to c} f(x) + g(x) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$.

(c) $\lim_{x\to c} f(x)g(x) = \Big(\lim_{x\to c} f(x) \Big) \Big(\lim_{x\to c} g(x) \Big)$.

(d) $\lim_{x\to c} f(x)/g(x) = \Big(\lim_{x\to c} f(x) \Big) / \Big(\lim_{x\to c} g(x) \Big)$ *so long as* $\lim_{x\to c} g(x) \neq 0$.

Proof. To prove (a) let $\lim_{x\to c} f(x) = L$. We should like to show that $\lim_{x\to c} bf(x) = bL$. From the the fact that $\lim_{x\to c} f(x) = L$ and the definition of the limit we know that $\forall \epsilon > 0 \exists \delta > 0 : |x - c| < \delta \implies |f(x) - L| < \epsilon$.

Fix $\epsilon > 0$, then

$$|bf(x) - bL| = |b||f(x) - L| \tag{2.11}$$

Now choose δ such that $|f(x) - L| < \epsilon/|b|$ (we can do this precisely because $\lim_{x\to c} f(x) = L$). Then

$$|bf(x) - bL| < |b|(\epsilon/|b|) = \epsilon \tag{2.12}$$

Thus we have proved that $\lim_{x\to c} bf(x) = bL = b\lim_{x\to c} f(x)$. $\square$

To prove (b) let $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Fix $\epsilon > 0$. We will use the triangle inequality, which says that $|x+y| \leq |x| + |y|$, to upper bound $|f(x)+g(x)-(L+M)|$ as follows:

$$|f(x)+g(x)-(L+M)| \leq |f(x)-L| + |g(x)-M| \tag{2.13}$$

Then since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} f(x) = M$ we may find δ_1 and δ_2 so that $|x-c| < \delta_1$ implies $|f(x)-L| < \epsilon/2$ and $|x-c| < \delta_2$ implies $|g(x)-M| < \epsilon/2$. Set $\delta = \min(\delta_1, \delta_2)$ then $|x-c| < \delta$ implies

$$|f(x)+g(x)-(L+M)| < \epsilon/2 + \epsilon/2 = \epsilon \tag{2.14}$$

Thus we have proved $\lim_{x\to c} f(x) + g(x) = L + M = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$. $\square$

To prove (c) we will have to be a bit clever and add and subtract the right thing inside $|f(x)g(x) - LM|$. Fix $\epsilon > 0$, then

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &\leq |f(x)g(x) - Lg(x)| + |Lg(x) - LM| \\ &= |g(x)||f(x) - L| + |L||g(x) - M| \end{aligned} \tag{2.15}$$

Now find δ_1 so that $|g(x)-M| < \min(M, \frac{\epsilon}{2L})$ whenever $|x-c| < \delta_1$. We can do this because $\lim_{x\to c} g(x) = M$. In this case $|g(x)| \leq 2M$ since $g(x) \in (0, 2M)$ and $|L||g(x)-M| < \frac{\epsilon}{2}$, thus:

$$|f(x)g(x) - LM| \leq 2M|f(x) - L| + \frac{\epsilon}{2} \tag{2.16}$$

Meanwhile if we find δ_2 so that $|f(x)-L| < \frac{\epsilon}{4M}$ (which we can do because $\lim_{x\to c} f(x) = L$) and set $\delta = \min(\delta_1, \delta_2)$ then for any $|x-c| < \delta$ we have

$$|f(x)g(x) - LM \leq 2M\frac{\epsilon}{4M} + \frac{\epsilon}{2} = \epsilon \tag{2.17}$$

Thus indeed $\lim_{x\to c} f(x)g(x) = LM = \left(\lim_{x\to c} f(x)\right)\left(\lim_{x\to c} g(x)\right)$. $\square$

To prove (d) we just need to prove that $\lim_{x\to c} g(x) = M$ implies $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ for $M \neq 0$, since if this is true then we can use (c) to write

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} f(x) \cdot \frac{1}{g(x)} = \lim_{x\to c} f(x) \lim_{x\to c} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M} \tag{2.18}$$

It remains to prove that $\lim_{x\to c} g(x) = M$ implies $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{M}$ when $M \neq 0$. Fix $\epsilon > 0$, then

$$|\frac{1}{g(x)} - \frac{1}{M}| = \frac{|g(x) - M|}{M|g(x)|} \tag{2.19}$$

Now find δ so that $|g(x)-M| < \min(\frac{M}{2}, \frac{M^2}{2}\epsilon)$ whenever $|x-c| < \delta$. Note that it is at this point that the proof would fail at $M = 0$, since we would be setting $\delta = \min(0,0) = 0$ which is not allowed. In this case the denominator in the above expression is lower bounded as $M|g(x)| \geq \frac{M^2}{2}$

since $|g(x)| \in (M/2, 3M/2)$, meanwhile the numerator is upper bounded by $|g(x) - M| \leq \frac{M^2}{2}\epsilon$. Thus

$$|\frac{1}{g(x)} - \frac{1}{M}| = \frac{|g(x) - M|}{M|g(x)|} < \frac{(M^2/2)\epsilon}{(M^2/2)} = \epsilon \tag{2.20}$$

This concludes the proof that $\lim_{x\to c}(f(x)/g(x)) = (\lim_{x\to c} f(x))/(\lim_{x\to c} g(x))$ whenever $\lim_{x\to c} g(x) \neq 0$. □ □

The wonderful thing about Theorem 2.1 is that now we can use it to compute a whole bunch of limits without resorting to $\epsilon - \delta$ calculus (fun as it is). But first, we will also need the concept of a continuous function. We already gave examples of the types of discontinuities in Figure 38, so intuitively a continuous function is simply a function that doesn't experience any of those types of discontinuities on its domain. But actually now that we have the limit, we can be precise about what we mean:

Definition. A function is called *continuous* at a point y in its domain if $\lim_{x\to y} f(x) = f(y)$. It is called continuous if it is continuous at every point in its domain.

For example, all polynomials and exponential functions are continuous, $\ln(x)$ is continuous on $(0, \infty)$, $\frac{1}{x}$ is continuous away from zero, $\tan(x)$ is continuous away from $\{\frac{\pi}{2} + \pi k : k \in \mathbb{Z}\}$, etc. As it turns out Theorem 2.1 immediately buys us analogous properties for continuity:

Theorem 2.2. *Let $f(x)$ and $g(x)$ be continuous functions at x_0. Then*

(a) $kf(x)$ is continuous at x_0 for any constant k.

(b) $f(x) + g(x)$ is continuous at x_0.

(c) $f(x)g(x)$ is continuous at x_0.

(d) $f(x)/g(x)$ is continuous at x_0 so long as $g(x_0) \neq 0$.

Proof. For example $\lim_{x\to x_0} kf(x) = k\lim_{x\to x_0} f(x) = kf(x_0) = (kf)(x_0)$. The proof is identical for $(b) - (d)$, just using the corresponding property of limits. □

The astute reader might note that the definition of a limit gives us the following $\epsilon - \delta$ definition of continuity:

$$f \text{ is continuous at } y \iff \forall \epsilon > 0 \exists \delta > 0 : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \tag{2.21}$$

We shall not use this definition in this course, however, preferring to leave it in the form $\lim_{x\to y} f(x) = f(y)$.

Example 2.5. Compute $\lim_{x\to\frac{\pi}{4}} \sin(x)\tan(x)$. Both $\sin(x)$ and $\tan(x)$ are continuous at $\pi/4$ and thus so is their product. We may therefore pass the limit inside to get $\sin(\pi/4)\tan(\pi/4) = \frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}}$.

3 Integrals and the Fundamental Theorem of Calculus

References

[HH+02] Gleason Hughes-Hallett et al. *Single and multivariable Calculus, John Wilet & Sons, Inc., New York, Chichester*. 2002.