Lipschitz Analysis of Noisy Quantum Inference as Phase Retrieval

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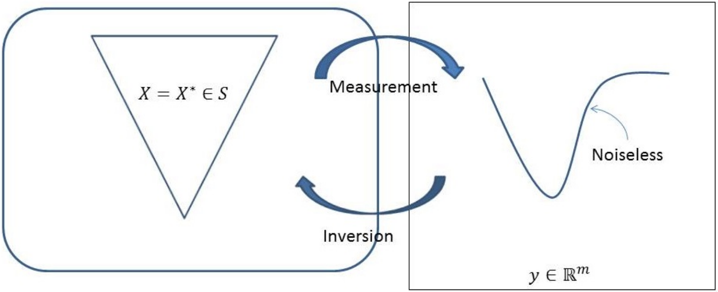
Overview

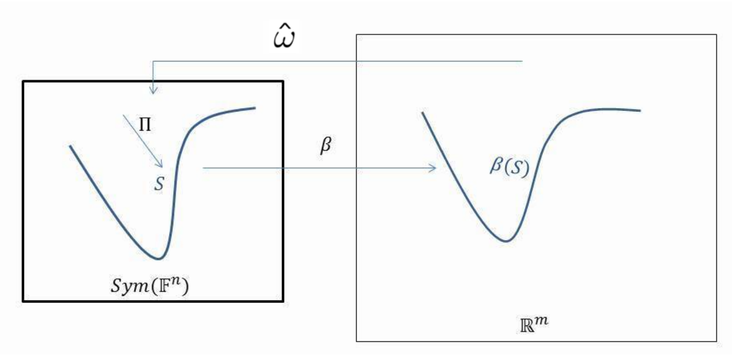
Lipschitz Inversion

Given: A nonlinear map $\beta : \mathcal{S} \to \mathbb{R}^m$ from a metric space $(\mathcal{S}, D)$ to Euclidean space $(\mathbb{R}^m, d)$. We also assume $\mathcal{S} \subset H$ where H is a Hilbert space.
Would like: A left inverse $\omega : \mathbb{R}^m \to \mathcal{S}$ that is globally Lipschitz.



Approach

1. Obtain an L-Lipschitz inverse $\beta^{-1} : \beta(\mathcal{S}) \to \mathcal{S} \subset H$
2. Use Kirszbraun's Theorem to obtain an L-Lipschitz extension $\hat{\omega} : \mathbb{R}^m \to H$. See recent constructible proofs of Kirszbraun [AGM18].
3. If $\mathcal{S}$ is a Lipschitz retract, form $\omega : \mathbb{R}^m \to \mathcal{S}$, $\omega = \Pi \circ \hat{\omega}$ where $\Pi : H \to \mathcal{S}$ is the Lipschitz retraction.



Spaces

In this talk we will take $H = \mathrm{Sym}(\mathbb{C}^n) \subset \mathbb{C}^{n\times n}$ to be our ambient Hilbert space, endowed with real inner product $\langle A, B\rangle_{\mathbb{R}} := \Re\mathrm{Tr}[A^* B]$. Options for $\mathcal{S}$ include

1. Convex cone of PSD

$$\mathrm{Sym}_{\mathbb{C}}^{+} := \{S \in \mathrm{Sym}(\mathbb{C}^n) | S \geq 0\}$$

2. Low rank quantum states

$$\mathcal{M}_r := \{S \in \mathrm{Sym}_{\mathbb{C}}^{+} | \mathrm{rank}(S) \leq r, \mathrm{Tr}[S] = 1\}$$

3. Pure quantum states $\mathcal{M}_1$
4. Cone of low-rank mixed signature signals

$$S^{p,q} := \{S \in \mathrm{Sym}(\mathbb{C}^n) | S \text{ has at most } p \text{ positive eigenvalues and } q \text{ negative eigenvalues}\}$$

$S^{p,q}$

We denote by $\mathring{S}^{p,q}$ the subset of $S^{p,q}$ having exactly p positive and q negative eigenvalues. One can show

Theorem

For every pair of non-negative integers p and q

- $S^{p,q}$ *is a closed semi-algebraic set.*
- $S^{p,q} = S^{p,0} + S^{0,q} = S^{p,0} - S^{q,0}$.
- $S^{p,q} \simeq \mathbb{C}^{n,p+q}/U(p,q)$ *where the quotient is by the $p+q \times p+q$ possibly indefinite unitary matrices acting on the right.*
- $S^{p,q} = \{xx^* - yy^* | x \in \mathbb{C}^{n,p} \quad y \in \mathbb{C}^{n,q}\}$
- $S^{p,q} = \cup_{0 \leq s \leq p} \cup_{0 \leq t \leq q} \mathring{S}^{s,t}$.
- $\mathring{S}^{p,q}$ *is a smooth manifold of dimension $2n(p+q) - (p+q)^2$.*
- $\mathring{S}^{r,0} \simeq \mathbb{C}^{n,r}_*/U(r)$ *where $\mathbb{C}^{n,r}_*$ denotes the full rank tall matrices.*
- $S^{r,r} \simeq T\mathring{S}^{r,0}$ *where $T\mathring{S}^{r,0}$ is the tangent bundle.*

Semi Metric Structure on $\mathbb{C}^{n,r}$ induced by $S^{r,0}$

The identification $S^{r,0} \simeq \mathbb{C}^{n,r}/U(r)$ can be made explicit via the quotient map

$$\pi : \mathbb{C}^{n,r} \to S^{r,0}$$
$$\pi(z) = zz^*$$

Given that, we find two non-equivalent classes of semi metrics on $\mathbb{C}^{n,r}$ $d_p, D_p : \mathbb{C}^{n,r} \times \mathbb{C}^{n,r} \to \mathbb{R}$.
The *norm induced metrics*:

$$d_p(x,y) = ||\pi(x) - \pi(y)||_p = ||xx^* - yy^*||_p$$

And the *natural metrics*:

$$D_p(x,y) = \min_{\substack{x\in[x] \\ y\in[y]}} ||x-y||_p = \min_{U\in U(r)} ||x - yU||_p$$

We have the following identity:

$$D_2(x,y) = \sqrt{\mathrm{Tr}(\pi(x)) + \mathrm{Tr}(\pi(y)) - 2||\sqrt{\pi(x)}\sqrt{\pi(y)}||_1}$$

Remark: it is a consequence of the Arithmetic-Geometric Mean Inequality that: [BK00]

$$\frac{1}{2}||\sqrt{\pi(x)} - \sqrt{\pi(y)}||_2^2 \leq \min_{\substack{x\in[x] \\ y\in[y]}} ||x-y||_2^2 \leq ||\sqrt{\pi(x)} - \sqrt{\pi(y)}||_2^2$$

That is D_2 is comparable to the Bures-Hellinger distance.

Quantum Tomography

It is common in physics to model a system as a statistical ensemble over pure quantum states $\psi_1, \ldots, \psi_r \subset \mathcal{H}$ having ensemble probabilities p_i of being in state ψ_i. In the finite dimensional case, we may take $\mathcal{H} = \mathbb{C}^n$. In this case, the density matrix

$$\rho := \sum_{j=1}^{r} p_r \psi_j \psi_j^*$$

contains all of the knowable information about the system. For instance, the expectation of a given observable $A \in \mathrm{Sym}(\mathbb{C}^n)$ is $\mathrm{Tr}[\rho A]$. Note that the collection of all such density matrices is precisely $\mathcal{M}_r$. The problem of quantum tomography is to infer ρ from noisy measurements of the form

$$\begin{bmatrix} \mathrm{Tr}[\rho F_1] \\ \vdots \\ \mathrm{Tr}[\rho F_m] \end{bmatrix} + \nu \overset{\omega}{\mapsto} \hat{\rho} \qquad \nu \sim \mathcal{N}(0, \sigma^2)$$

In such a way that $||\rho - \hat{\rho}||_H \leq C||\nu||_2$

α and β maps

It suffices to consider our observables $\{F_k\}_{k=1}^m$ to lie in $\mathrm{Sym}(\mathbb{C}^n)^+$; if not we may simply define $\tilde{F}_k = F_k + \mu\mathbb{I}$ so that $\mathrm{Tr}[\rho F_k] = \mathrm{Tr}[\rho \tilde{F}_k] - \mu$ with $\mu \in \mathbb{R}$ large enough that all of the $\tilde{F}_k$ are positive. In this case there exists $z \in \mathbb{C}^{n,r}$ and $f_k \in \mathbb{C}^{n,r}$ so that that $\rho = \pi(z)$ and $\tilde{F}_k = \pi(f_k)$, so that the problem of noisy quantum inference is equivalent to whether the following map is Lipschitz invertible:

$$\beta : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m$$
$$\beta_k(z) := \langle \pi(z), \pi(f_k)\rangle_{\mathbb{R}} \qquad \text{(equal to } |\langle z, f_k\rangle_{\mathbb{C}}|^2 \text{ when } r = 1)$$

In analogy with the classical phase retrieval problem we also define

$$\alpha : \mathbb{C}^{n\times r}/U(r) \to \mathbb{R}^m$$
$$\alpha_k(z) := \langle \pi(z), \pi(f_k)\rangle_{\mathbb{R}}^{\frac{1}{2}} \qquad \text{(equal to } |\langle z, f_k\rangle_{\mathbb{C}}| \text{ when } r = 1)$$

Note that we are relaxing our requirement that the estimate $\hat{\rho} = \omega(x)$ have unit trace. We do this because $\mathcal{M}_r$ is not contractible when $r < n$, and so no Lipschitz retract $\Pi : Sym(\mathbb{C}^n) \to \mathcal{M}_r$ is possible.

α and β maps (stability)

The set of observables $\mathcal{F} := \{F_k\}_{k=1}^m = \{\pi(f_k)\}_{k=1}^m$ is called phase retrievable if the analysis maps α and β are injective.
By scaling, it is natural to analyze the Lipschitz constants of

$$\alpha : (\mathbb{C}^{n\times r}/U(r), D_p) \to (\mathbb{R}^m, ||\cdot||_2)$$
$$\beta : (\mathbb{C}^{n\times r}/U(r), d_p) \to (\mathbb{R}^m, ||\cdot||_2)$$

What we would like to show is the following:

Theorem

Assume $\mathcal{F} = \{F_1, \ldots, F_k\} \subset Sym_{\mathbb{C}}^+$ is phase retrievable. Then there are constants $a_0, A_0, b_0, B_0 > 0$ so that for every $x, y \in \mathbb{C}^{n\times r}/U(r)$

$$A_0 D_2(x,y)^2 \leq \sum_{k=1}^m |\langle \pi(x), \pi(f_k)\rangle_{\mathbb{R}}^{1/2} - \langle \pi(y), \pi(f_k)\rangle_{\mathbb{R}}^{1/2}|^2 \leq B_0 D_2(x,y)^2$$
$$a_0 d_1(x,y)^2 \leq \sum_{k=1}^m |\langle \pi(x), \pi(f_k)\rangle_{\mathbb{R}} - \langle \pi(y), \pi(f_k)\rangle_{\mathbb{R}}|^2 \leq b_0 d_1(x,y)^2$$

Remark: The nuclear norm is the easiest to manipulate in this context, but of course d_1 and d_2 are comparable.

Local Lipschitz Constants

In order to make the problem more tractable we analyze the local Lipschitz properties of α and β:

$$A(z) = \lim_{R\to 0} \inf_{\substack{D_2(x,z)<R \\ D_2(y,z)<R \\ \pi(x)\neq\pi(y)}} \frac{\sum_{k=1}^{m} |\langle \pi(x), \pi(f_k)\rangle_{\mathbb{R}}^{1/2} - \langle \pi(y), \pi(f_k)\rangle_{\mathbb{R}}^{1/2}|^2}{D_2(x,y)^2}$$

$$B(z) = \lim_{R\to 0} \sup_{\substack{D_2(x,z)<R \\ D_2(y,z)<R \\ \pi(x)\neq\pi(y)}} \frac{\sum_{k=1}^{m} |\langle \pi(x), \pi(f_k)\rangle_{\mathbb{R}}^{1/2} - \langle \pi(y), \pi(f_k)\rangle_{\mathbb{R}}^{1/2}|^2}{D_2(x,y)^2}$$

$$a(z) = \lim_{R\to 0} \inf_{\substack{d_1(x,z)<R \\ d_1(y,z)<R \\ \pi(x)\neq\pi(y)}} \frac{\sum_{k=1}^{m} |\langle \pi(x), \pi(f_k)\rangle_{\mathbb{R}} - \langle \pi(y), \pi(f_k)\rangle_{\mathbb{R}}|^2}{d_1(x,y)^2}$$

$$b(z) = \lim_{R\to 0} \sup_{\substack{d_1(x,z)<R \\ d_1(y,z)<R \\ \pi(x)\neq\pi(y)}} \frac{\sum_{k=1}^{m} |\langle \pi(x), \pi(f_k)\rangle_{\mathbb{R}} - \langle \pi(y), \pi(f_k)\rangle_{\mathbb{R}}|^2}{d_1(x,y)^2}$$

Realification

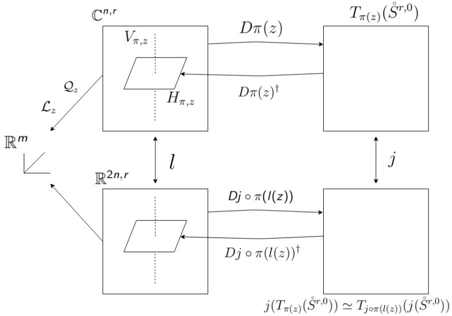
Because $D\pi(z) : \mathbb{C}^{n,r} \to T_{\pi(z)}(\mathring{S}^{r,0}), D\pi(z)(w) = zw^* + wz^*$ is real linear but not complex linear, it is natural to view the local Lipschitz problem in terms of the realifications of the objects involved. Define the linear isomorphism $I : \mathbb{C}^{n,r} \to \mathbb{R}^{2n,r}$ with $I(A) = \begin{bmatrix} \Re A \\ \Im A \end{bmatrix}$ and the algebra homomorphism $j : \mathbb{C}^{n,r} \to \mathbb{R}^{2n,2r}$ with $j(A) = \begin{bmatrix} \Re A & -\Im A \\ \Im A & \Re A \end{bmatrix}$. Note that

$$j(A) = \begin{bmatrix} I(A) & JI(A) \end{bmatrix} \qquad , J = \begin{bmatrix} 0 & -\mathbb{I}_{n\times n} \\ \mathbb{I}_{n\times n} & 0 \end{bmatrix}$$

We have, for example in the case $r = 1$:

$$\text{span}_{\mathbb{R}}\{iz\} = \text{Ker}(D\pi(z)) \simeq \text{Ker}(Dj \circ \pi(I(z))) = \text{span}(JI(z))$$

Sketch of argument



- For $z \in \mathbb{C}^{n,r}_*$ formulate $a(z)$ and $A(z)$ as

$$A(z) = \min_{\substack{w \in \mathbb{C}^{n,r} \\ ||w||_2 = 1}} ||\mathcal{L}_z \mathbb{P}_{H_{\pi,z}} w||_2, \quad a(z) = \min_{\substack{w \in T_\pi(z)(\mathring{S}^{r,0}) \\ ||w||_2 = 1}} ||\mathcal{L}_z D\pi(z)^\dagger w||_2$$

  for some linear operators $\mathcal{L}_z$ and $\mathcal{Q}_z$.
- Show that Ker$\mathcal{Q}_z$= Ker$\mathcal{L}_z$ = Ker$(D\pi(z))^\perp$ is exactly phase retrievability.
- Argue by contradiction that this implies $a_0, A_0 > 0$.

Known results for r=1: Phase retrievability

Theorem (B13)

Let $\mathcal{F}$ be a frame for $\mathbb{C}^n$. The following are equivalent

- $\mathcal{F}$ *is phase retrievable.*
- $\pi(Ker(\alpha)) \cap (S^{1,0} - S^{0,1}) = \pi(Ker(\alpha)) \cap (T\mathring{S}^{1,0}) = \{0\}$
- $span_{\mathbb{R}}\{f_k f_k^* z\}_{1 \leq k \leq m} = span_{\mathbb{R}}(iz)^{\perp}$ *for all* $z \in \mathbb{C}^n \setminus \{0\}$.
- $\dim span_{\mathbb{R}}\{f_k f_k^* z\}_{1 \leq k \leq m} \geq 2n - 1$ *for all* $z \in \mathbb{C}^n \setminus \{0\}$.

Note: If we define $\phi_k = l(f_k)$ then set $\Phi_k = j(f_k f_k^*) = \phi_k \phi_k^T + J\phi_k \phi_k^T J^T$, then we obtain two additional equivalent criteria via realification:

- $\text{span}\{\Phi_k l(z)\} = \text{span}\{Jl(z)\}^{\perp}$ for all $z \in \mathbb{C}^n \setminus \{0\}$
- $\dim \text{span}\{\Phi_k l(z)\} \geq 2n - 1$ for all $z \in \mathbb{C}^n \setminus \{0\}$.

Known results for r=1: Lipschitz inversion of α

Set $\Phi_k = j(f_k f_k^*) = \phi_k\phi_k^T + J\phi_k\phi_k^T J^T$ as before. For $z \in \mathbb{C}^n \setminus \{0\}$ define the real $2n \times 2n$ matrix $\mathcal{S}_z = \sum_{k:\Phi_k l(z)\neq 0} \frac{1}{\langle \Phi_k l(z), l(z)\rangle} \Phi_k l(z) l(z)^T \Phi_k$. Set $\mathcal{S}_0 = 0$. Then

Theorem (B13)

Let $\mathcal{F}$ be a phase retrievable frame for $\mathbb{C}^n$. Then

- *For every* $z \in \mathbb{C}^n \setminus \{0\}, A(z) = \lambda_{2n-1}(\mathcal{S}_z) > 0$
- *For every* $z \in \mathbb{C}^n \setminus \{0\}, \mathcal{S}_z \geq A(z)\mathbb{P}_{Jl(z)^\perp} = A(z)\mathbb{P}_{Ker Dj\circ\pi(l(z))^\perp}$
- $A_0 = A(0) > 0$
- $B(z) = \lambda_1(\mathcal{S}_z + \sum_{k:\langle z, f_k\rangle_{\mathbb{C}}=0} \Phi_k)$
- $B_0 = B(0) < \infty$

Remark: $\Phi_k l(z) = j(f_k f_k^* z) = j(\langle z, f_k\rangle_{\mathbb{C}} f_k)$. Hence $\Phi_k l(z) = 0 \iff \langle z, f_k\rangle_{\mathbb{C}} = 0$

Known results for r=1: Lipschitz inversion of β

Set $\Phi_k = j(f_k f_k^*) = \phi_k \phi_k^T + J\phi_k \phi_k^T J^T$. For $z \in \mathbb{C}^n \setminus \{0\}$ define the real $2n \times 2n$ matrix $\mathcal{R}_z = \sum_{k=1}^m \Phi_k l(z) l(z)^T \Phi_k$. Then

Theorem (B13)

Let $\mathcal{F}$ be a phase retrievable frame for $\mathbb{C}^n$. Then

- *For every $z \in \mathbb{C}^n \setminus \{0\}$, $a(z) = \lambda_{2n-1}(\mathcal{R}_z)/||l(z)||_2^2$*
- *For every* $z \in \mathbb{C}^n \setminus \{0\}, \mathcal{R}_z \geq a(z)||l(z)||_2^2 \mathbb{P}_{Jl(z)^\perp} = a(z)||l(z)||_2^2 \mathbb{P}_{Ker Dj \circ \pi(l(z))^\perp}$
- $a_0 = a(0) = \min_{||z||_2=1} \lambda_{2n-1}(\mathcal{R}_z) > 0$
- *For every $z \in \mathbb{C}^n \setminus \{0\}$, $b(z) = \lambda_1(\mathcal{R}_z)/||l(z)||^2$*
- $b_0 = b(0) < \infty$

Computational lemma concerning d_p, D_p

The following facts are key in proving stability of α and β respectively:

Lemma

- $D_p(x,y) = ||x-y||_p$ *if and only if* $x^*y = y^*x$ *and* $x^*y \geq 0$
- $d_p(x,y) = ||D\pi(\frac{x+y}{2})(x-y)||_p$ *Where* $D\pi(z) : \mathbb{C}^{n,r} \to T_{\pi(z)}(\mathring{S}^{r,0})$ *is the differential of* π. *Moreover, when* $r = 1$ *we have*

$$d_1(x,y) = ||xx^* - yy^*||_1 = ||\frac{x+y}{2}||_2 ||\mathbb{P}_{KerD\pi(\frac{x+y}{2})^\perp}(x-y)||_2$$

Theorem

$S^{r,0}$ is a disjoint union of smooth manifolds $\mathring{S}^{s,0}$, each the image of the Riemannian submersion $\pi : \mathbb{C}_*^{n,s} \to \mathring{S}^{s,0}$. That is to say if $D\pi(z) : \mathbb{C}^{n,r} \to T_{\pi(z)}(S^{r,0})$ is the differential of π and $\mathbb{C}^{n,r} = H_z \oplus V_z$ is the decomposition into the horizontal and vertical space, then $D\pi(z)|_{H_z}$ is a metric preserving surjection for every $z \in \mathbb{C}_*^{n,r}$. Moreover,

- $V_z = KerD\pi(z) = \{izS | S \in Sym(\mathbb{C}^n)\}$. *Since* $z \in \mathbb{C}_*^{n,r}$ *we have* $\dim_{\mathbb{R}} V_z = r^2$
- $H_z = (KerD\pi(z))^{\perp} = \{Hz + Rz | H \in Sym(\mathbb{C}^n), Ran(H) \subset Ran(z), Ran(R) \perp Ran(z)\}$. *Since* $z \in \mathbb{C}_*^{n,r}$ *we have* $\dim_{\mathbb{R}} H_z = 2nr - r^2$.
- *The Riemannian submersion π induces a unique Riemannian metric on $\mathring{S}^{r,0}$ with* $g_{\pi(z)}(X_1, X_2) = \langle D\pi(z)^{\dagger} X_1, D\pi(z)^{\dagger} X_2 \rangle_{\mathbb{R}}$. *This metric generates a geodesic distance which is precisely D_2, and can be written explicitly as*

$$g_{\pi(z)}(X_1, X_2) = Tr[\int_0^{\infty} X_1 \mathbb{P}_{Ran(z)} e^{-\pi(z)u} X_2 \mathbb{P}_{Ran(z)} e^{-\pi(z)u} du] + \Re Tr[\mathbb{P}_{Ran(z)^{\perp}} X_1 \pi(z)^{\dagger} X_2]$$

As before we can lift to the realification, and after a bit of work obtain

$$\mathrm{Ker}(Dj \circ \pi(l(z))) = \{Jl(z)A | A \in \mathrm{Sym}(\mathbb{R}^r)\} \oplus \{l(z)K | K \in \mathrm{Asym}(\mathbb{R}^r)\}$$

Geodesics of $\mathring{S}^{r,0}$

Following [BTY18] one can employ the following theorem:

Theorem

Let $(\mathcal{M}, h)$ and $(\mathcal{N}, g)$ be Riemannian manifolds and $\pi : (\mathcal{M}, h) \to (\mathcal{N}, g)$ a Riemannian submersion. Let γ be a geodesic in $(\mathcal{M}, h)$ such that $\gamma'(0)$ is horizontal. Then

- *$\gamma'(t)$ is horizontal for all t.*
- *$\pi \circ \gamma$ is a geodesic in $(\mathcal{N}, g)$ of the same length as γ*

To obtain the geodesic connecting $A, B \in (\mathring{S}^{r,0}, g)$ as

$$\gamma_{A,B} : [0,1] \to \mathring{S}^{r,0}$$
$$\gamma(t) = t^2 B + (1-t)^2 A + t(1-t)(\sqrt{AB} + \sqrt{BA})$$

The length of this geodesic is $D_2(a, b)$ where $\pi(a) = A$ and $\pi(b) = B$.

Results for $r > 1$: Phase retrievability

Let $\mathcal{F}$ be a frame for $\mathbb{C}^n$. The following are equivalent

Theorem

- $\mathcal{F}$ *is phase retrievable*
- $\pi(Ker(\alpha)) \cap (S^{r,0} - S^{r,0}) = \pi(Ker(\alpha)) \cap (T\mathring{S}^{r,0}) = 0$
- $span_{\mathbb{R}}\{f_k f_k^* z\} = \{izS | S \in Sym(\mathbb{C}^r)\}^{\perp}$
- $span\{\Phi_k I(z)\} = (\{JI(z)A | A \in Sym(\mathbb{R}^r)\} \oplus \{I(z)K | K \in Asym(\mathbb{R}^n)\})^{\perp}$ *for all* $z \in \mathbb{C}_*^{n,r}$
- $\dim span_{\mathbb{R}}\{f_k f_k^* z\} \geq 2nr - r^2$ *for all* $z \in \mathbb{C}_*^{n,r}$.

Results for $r > 1$: Lipschitz inversion of α

Define the $2nr \times 2nr$ matrices

$$\mathbb{F}_k = \left[\begin{array}{c|c|c} \Phi_k & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \Phi_k \end{array}\right] = \Phi_k \otimes \mathbb{I}_{r,r}$$

$$\mathcal{S}_z = \sum_{k:\Phi_k I(z) \neq 0} \frac{1}{\langle \Phi_k I(z), I(z) \rangle} \mathbb{F}_k \left[\begin{array}{c} I(z^1) \\ \hline \vdots \\ \hline I(z^r) \end{array}\right] \left[\begin{array}{c} I(z^1) \\ \hline \vdots \\ \hline I(z^r) \end{array}\right]^T \mathbb{F}_k \qquad \mathcal{T}_z = \mathcal{S}_z + \sum_{k:\Phi_k I(z)=0} \mathbb{F}_k$$

Theorem

Let $\mathcal{F}$ be a phase retrievable frame for $\mathbb{C}^{n,r}$. Then

- *For every* $z \in \mathbb{C}_*^{n,r}$, $A(z) = \min_{||w||_2=1} \sum_{k:\Phi_k I(z) \neq 0}^m Tr[I(z)\Phi_k \mathbb{P}_{Ker(Dj\circ\pi(I(z)))^\perp} I(w)]^2 = \lambda_{2nr-r^2}(\mathcal{S}_z) > 0$
- $\mathcal{S}_Z \geq A(z)\mathbb{P}_{(\{JI(z)A | A \in Sym(\mathbb{R}^r)\} \oplus \{I(z)K | K \in Asym(\mathbb{R}^n)\})^\perp} = A(z)\mathbb{P}_{Ker(Dj\circ\pi(I(z)))^\perp}$
- $B(z) = \max_{||w||_2=1} \sum_{k:\Phi_k I(z) \neq 0}^m Tr[I(z)\Phi_k I(w)]^2 + \sum_{k:\Phi_k I(z)=0} Tr[I(w)^T \Phi_k I(w)] = \lambda_1(\mathcal{T}_z)$
- $A_0 = A(0) > 0$ *and* $B_0 = B(0) < \infty$

To be continued...

- Analagous results for Lipschitz inversion of β.
- Relation of local Lipschitz constants to frame constants
- Determine good Lipschitz retract $\Pi : \mathrm{Sym}(\mathbb{C}^n) \to S^{r,0}$ and $\mathrm{Lips}(\Pi)$.

Thank you!

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